Uncertainty Characterization via Scenarios

Uncertainty Characterization via Scenarios

- A continues stochastic process: If we denote the wind speed in hour t at a given site A by v_t^A, then the set of random variables $\boldsymbol{V}^A = \{v_t^A, t = 1, 2, \ldots, 24\}$ describing wind speed at site. A throughout a day is a continuous stochastic process.

- A discrete stochastic process: Consider the binary variable u_{it}, which is equal to 1 if generating unit *i* is available in hour t and o otherwise. The collection of random variables $\boldsymbol{U}_i = \{u_{it}, t = 1, 2, \ldots, 168\}$ representing the availability of unit i throughout a week is a discrete stochastic process.
- Note that a discrete stochastic process can be represented by a finite set of actual vectors, referred to as scenarios, resulting from the combinations of all the discrete values that its component random variables can adopt.
- In mathematical terms, if $\boldsymbol{\lambda}$ is a discrete stochastic process, it can be expressed as $\boldsymbol{\lambda} = \{\boldsymbol{\lambda}(\omega), \omega = 1, 2, \ldots, N_\Omega\}$ where ω is the scenario index and N_Ω is the number of possible scenarios. In order for the discrete stochastic process to be perfectly determined, a probability of occurrence $\pi(\omega)$ needs to be associated with each realization $\boldsymbol{\lambda}(\omega)$ such that $\sum_{\omega=1}^{N_\Omega} \pi(\omega) = 1$.

Uncertainty Characterization via Scenarios

Example (Mathematical representation of a discrete stochastic process)

Let us consider the discrete stochastic process $\boldsymbol{U}_i$ representing the availability of generating unit i throughout a time horizon comprising two hourly periods. Process $\boldsymbol{U}_i$ can be mathematically expressed as $\boldsymbol{U}_i = \{\boldsymbol{U}_i(\omega) = [u_{i1}(\omega), u_{i2}(\omega)], \omega = 1, \ldots, 4\}$
where binary variable $u_{it}(\omega), t = 1, 2,$ is equal to 1 if unit *i* is available in time period *t* and scenario ω, and 0 otherwise. Specifically, each scenario $\boldsymbol{U}_i(\omega)$ can be written in the form

$$
\begin{aligned}
\boldsymbol{U}_i(1) &= [1, 1], \text{ with a probability of occurrence } \pi(1) = 0.5 \\
\boldsymbol{U}_i(2) &= [0, 1], \text{ with } \pi(2) = 0.2 \\
\boldsymbol{U}_i(3) &= [1, 0], \text{ with } \pi(3) = 0.2 \\
\boldsymbol{U}_i(4) &= [0, 0], \text{ with } \pi(4) = 0.1.
\end{aligned}
$$

Uncertainty Characterization via Scenarios

Solving optimization problems where uncertainty on input data is modeled by continuous stochastic processes is very difficult, or even impossible in many cases. In contrast, discrete stochastic processes can be easily embedded into an optimization problem using the deterministic equivalent problem.

- For this reason, prior to finding the solution of a stochastic programming problem, every continuous stochastic process is to be replaced by an approximate discrete one.

Uncertainty Characterization via Scenarios

Example (Approximation of a continuous process with a discrete one)

After a comprehensive statistical analysis, we come to the conclusion that the continuous stochastic process $\boldsymbol{V}^A$, representing wind speed at site A throughout a time horizon of two hours, can be well approximated by the discrete process $\boldsymbol{\nu}^A = \{\boldsymbol{\nu}^A(\omega) = [v_1^A(\omega), v_2^A(\omega)], \omega = 1, 2, 3\}$, where variable $v_t^A(\omega), t = 1, 2$, represents wind speed (in m/s) at site A in period *t* and scenario *ω*. Specifically, each scenario $\boldsymbol{\nu}^A(\omega)$ is given by

$$\begin{aligned}
\boldsymbol{\nu}^A(1) &= [5.5, 7.8] \text{ m/s, with a probability of occurrence } \pi(1) = 0.4 \\
\boldsymbol{\nu}^A(2) &= [10.0, 13.7] \text{ m/s, with } \pi(2) = 0.3 \\
\boldsymbol{\nu}^A(3) &= [1.2, 2.4] \text{ m/s, with } \pi(3) = 0.3.
\end{aligned}$$

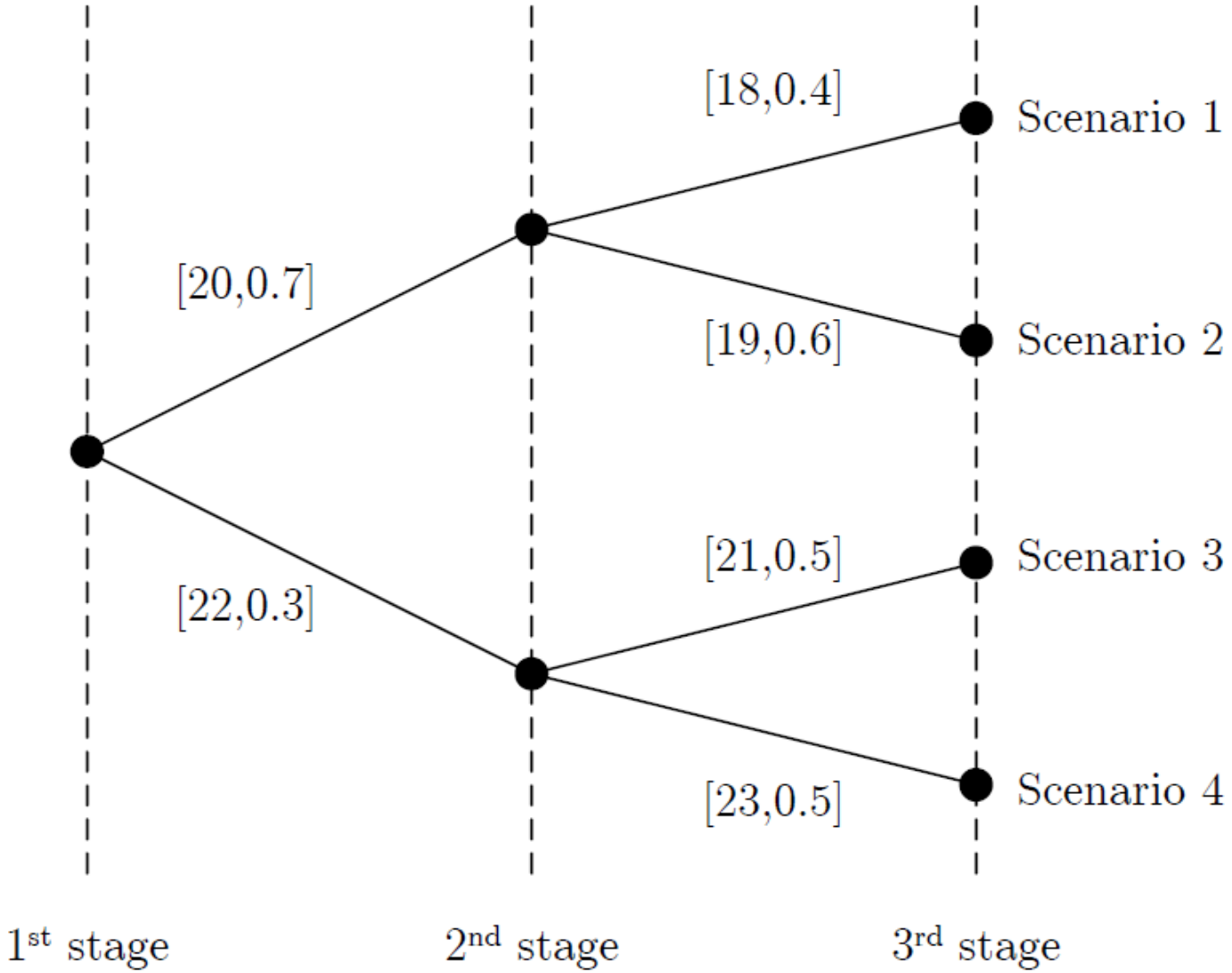
- ✓ A good discrete approximations often requires the generation of a large number of scenarios.

Uncertainty Characterization via Scenarios

❑ Scenario Generation

- In stochastic programming, stochastic processes can be represented using continuous or discrete random variables.
- In the best case, stochastic programming problems with continuous random variables can only be solved in small or illustrative instances.
- For this reason, the discrete representation of random variables using a finite set of possible outcomes becomes indispensable in actual decision-making problems under uncertainty.
- The set of finite values used to model a random variable is usually arranged in a so-called *scenario tree*.
- Each single path between the root node and a leave is named scenario.
- In a scenario tree, the arcs represent different realizations of the random variables. Each arc has associated a probability of occurrence. In this way, the probability of a scenario is the product of all arcs probabilities associated with that scenario.

Uncertainty Characterization via Scenarios



Uncertainty Characterization via Scenarios

- Different techniques have been proposed in the technical literature to build scenario trees. Some relevant methods are the following:

I. Generation of data trajectories or path-based methods
II. Moment matching
III. Internal sampling
IV. Scenario reduction

✓ In this book, a procedure combining both path-based methods and scenario reduction techniques is used to generate scenario trees.
✓ A large enough number of scenarios is first generated by ARIMA models.

Uncertainty Characterization via Scenarios

❑ Scenario Generation using ARIMA Models

The probabilistic structure of a stochastic process $\boldsymbol{Y}$ is determined by identifying the joint distribution of its random variables $\{y_t, t \in T\}$.

Roughly speaking, this distribution explains both the probabilistic behavior of each random variable y_t on its own (marginal distributions), and the interrelations existing among all of them (statistical dependencies).

In practice, the determination of the joint distribution is usually a complex and cumbersome endeavor. However, this estimation becomes much easier under the following two assumptions:

1. The joint distribution is a multivariate Gaussian distribution and therefore, is determined by just specifying the mean vector and the variance-covariance matrix of the random variables constituting the stochastic process.
2. The stochastic process under analysis is stationary, which means that neither the mean vector nor the variance-covariance matrix depend on time *t*.

Uncertainty Characterization via Scenarios

- With reference to assumption 1, the non-diagonal elements of a variance-covariance matrix are generally referred to as *autocovariances*. Through a simple standardization process consisting in just dividing the *autocovariances* by the corresponding product of standard deviations, the *autocorrelations* are obtained.

- The time series theory based on autoregressive moving average (ARMA) models relies on these two premises . An ARMA(p, q) process Y is mathematically expressed as

$$y_t = \sum_{j=1}^{p} \phi_j y_{t-j} + \varepsilon_t - \sum_{j=1}^{q} \theta_j \varepsilon_{t-j}$$

With p autoregressive parameters $\phi_1, \phi_2, \ldots, \phi_p$ and q moving average parameters $\theta_1, \theta_2, \ldots, \theta_q$. The term ε_t in equation stands for an uncorrelated normal stochastic process with mean zero and variance σ_ε^2 , and is also uncorrelated with $y_{t-1}, y_{t-2}, \ldots, y_{t-p}$.
Stochastic process ε_t is also referred to as white noise, innovation term, or error term.

Uncertainty Characterization via Scenarios

The procedure to generate a set of scenarios for stochastic process Y is based on the sampling of the error terms from their distribution $\varepsilon_t \sim N(0, \sigma)$.

❑ Algorithm

- *Step 1*: Initialize the scenario counter: $\omega \leftarrow 0$.
- *Step 2*: Update the scenario counter and initialize the time period counter: $\omega \leftarrow \omega + 1$, $t \leftarrow 0$.
- *Step 3*: Update the time period counter: $t \leftarrow t + 1$.
- *Step 4*: Randomly generate $\varepsilon_t \sim \mathrm{N}(0, \sigma)$.
- *Step 5*: Evaluate expression (3.1) to obtain $y_{t\omega}$.
- *Step 6*: If $t < N_{\mathrm{T}}$ go to Step 3), else go to Step 7)
- *Step 7*: If $\omega < N_{\Omega}$ go to Step 2), else the scenario-generation process concludes.

Uncertainty Characterization via Scenarios

Example (Scenario generation based on ARMA models)

Consider the following ARMA(1,1) model for characterizing a certain variable y_t

$$y_t = \phi_1 y_{t-1} + \varepsilon_t - \theta_1 \varepsilon_{t-1}.$$

The parameters of the model are $\phi_1 = 0.9914$ and $\theta_1 = 0.1131$. The value of y_0 is 21.3567 and $\varepsilon_0 = -0.0124$. The standard deviation of the error term is 0.021. The aim of this example is to generate a set of two scenarios for a three-period horizon. The resulting set of scenarios is denoted as

$$\{y_{t\omega},\ t = 1, 2, 3; \omega = 1, 2\}$$

Uncertainty Characterization via Scenarios

To generate scenario $\omega = 1$

$$
\begin{aligned}
y_{11} &= \phi_1 y_0 + \varepsilon_{11} - \theta_1 \varepsilon_0 \\
&= 0.9914 \times 21.3567 + 0.0060 - 0.1131 \times (-0.0124) = 21.1804 \\
y_{21} &= \phi_1 y_{11} + \varepsilon_{21} - \theta_1 \varepsilon_{11} \\
&= 0.9914 \times 21.1804 - 0.0241 - 0.1131 \times 0.0060 = 20.9735 \\
y_{31} &= \phi_1 y_{21} + \varepsilon_{31} - \theta_1 \varepsilon_{21} \\
&= 0.9914 \times 20.9735 + 0.0250 - 0.1131 \times (-0.0241) = 20.8209.
\end{aligned}
$$

Uncertainty Characterization via Scenarios

To generate scenario $\omega = 2$

$$
\begin{aligned}
y_{12} &= \phi_1 y_0 + \varepsilon_{12} - \theta_1 \varepsilon_0 \\
&= 0.9914 \times 21.3567 + 0.0250 - 0.1131 \times (-0.0124) = 21.1994 \\
y_{22} &= \phi_1 y_{12} + \varepsilon_{22} - \theta_1 \varepsilon_{12} \\
&= 0.9914 \times 21.1994 - 0.0008 - 0.1131 \times 0.0250 = 21.0135 \\
y_{32} &= \phi_1 y_{22} + \varepsilon_{32} - \theta_1 \varepsilon_{22} \\
&= 0.9914 \times 21.0135 + 0.0069 - 0.1131 \times (-0.0008) = 20.8398.
\end{aligned}
$$

Uncertainty Characterization via Scenarios

Questioning the stationarity assumption. Integrated and seasonal ARMA processes

One of the major hypotheses concerning the use of ARMA models is to assume that the stochastic process under study is stationary. That is, the mean and the variance of the process must remain stable throughout the time.

However, this property does not always hold and it is necessary to make some transformations to the process in order to achieve stationarity.

In order to attain stationarity for the variance, the so-called Box-Cox transformations are widely used. Basically, these transformations consist in applying either the logarithm or the square root functions to the original process.

Uncertainty Characterization via Scenarios

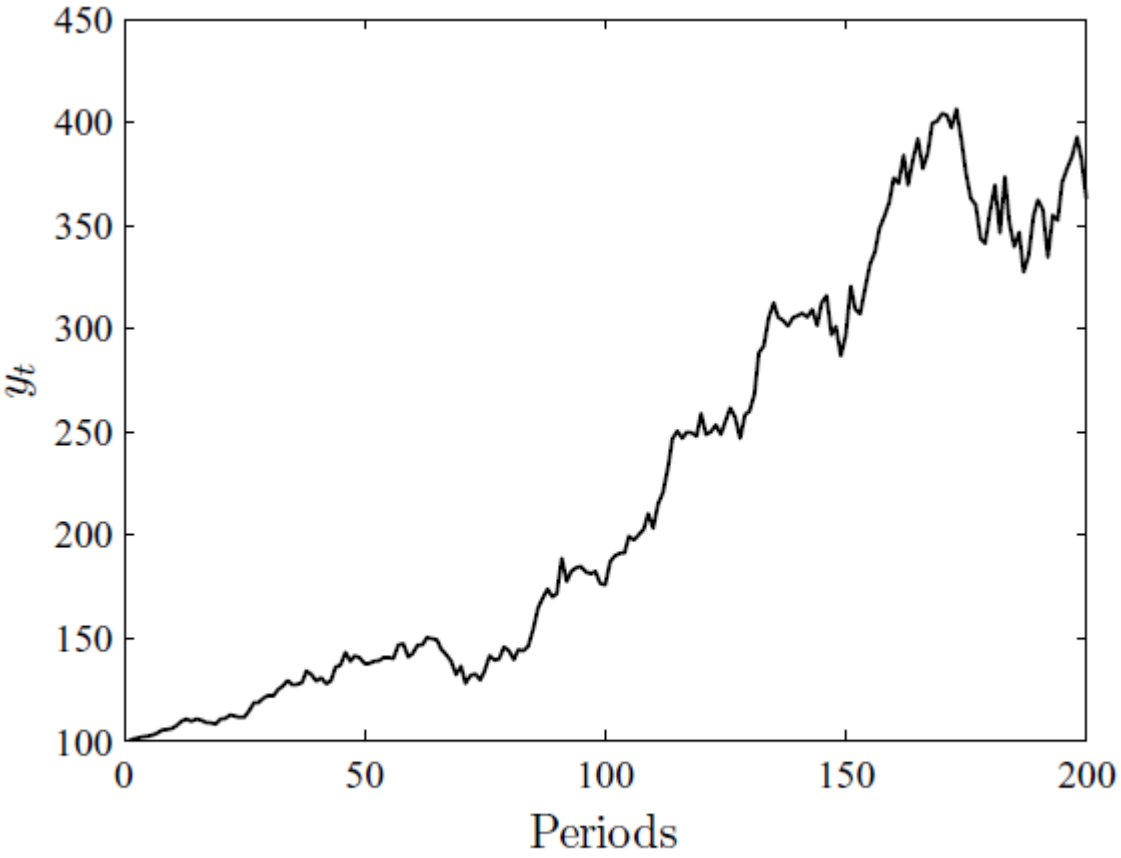


Fig. 3.2 Non-stationary process.

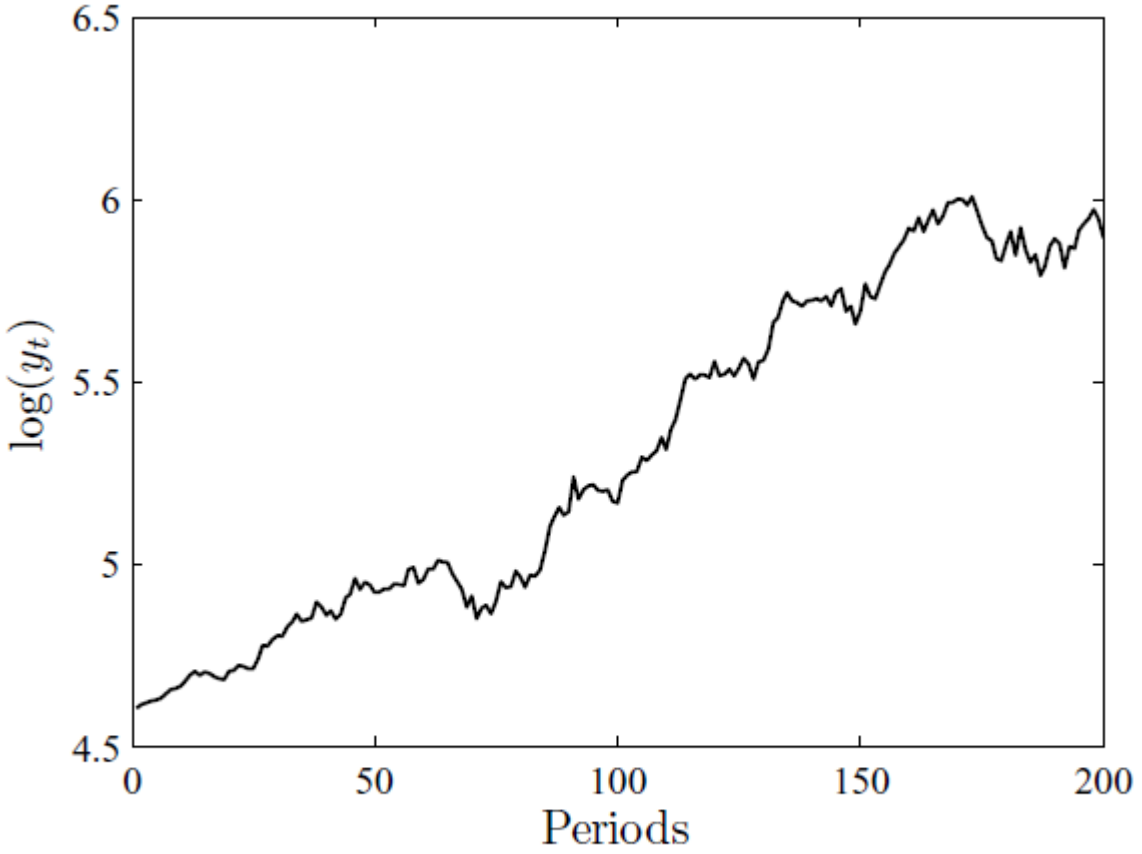


Fig. 3.3 Logarithm transformation of the non-stationary process.

Uncertainty Characterization via Scenarios

- The typical manner of obtaining a stable process in terms of the mean is to differentiate the process. For instance, differencing process y_t with order 1, denoted as $\nabla^1 y_t$ is expressed as

$$\nabla^1 y_t = (1 - B) y_t = y_t - y_{t-1},$$

where B is the backshift operator, i.e., $B^j y_t = y_{t-j}$.

Usually, a difference order equal to 1 is enough to obtain a stable mean.
In general, differencing a process with order *d* is indicated as $\nabla^d y_t = (1 - B)^d y_t$.

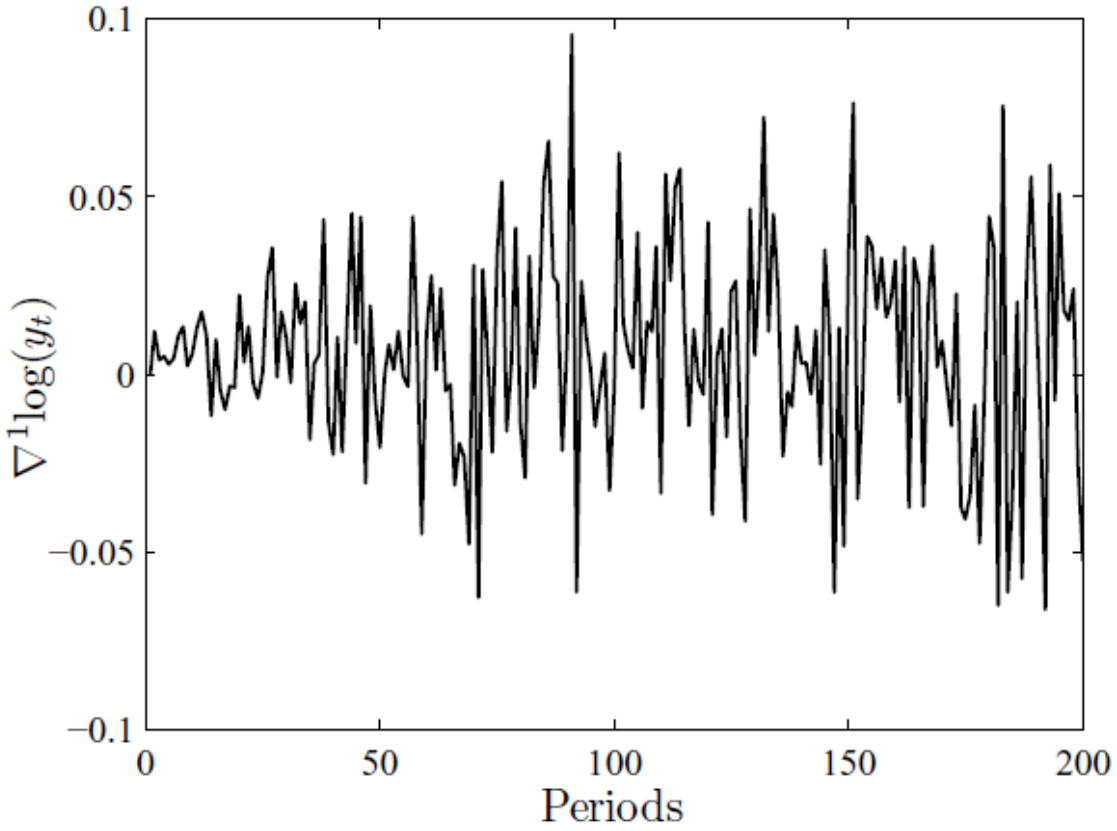


Fig. 3.4 Differentiation of the non-stationary process.

Uncertainty Characterization via Scenarios

This differencing procedure together with the ARMA models previously presented gives rise to ARIMA models for non-stationary time series. ARIMA stands for autoregressive integrated moving average, where integrated indicates that the original non-stationary process is obtained by integrating the stationary process resulting after differencing.

In this way, ARIMA models are defined by three parameters (p, d, q) corresponding to the number of autoregressive terms, the differencing order, and the number of moving-average terms, respectively. The general expression of an ARIMA model with parameters (p, d, q) is

$$\left(1-\sum_{j=1}^{p}\phi_j B^j\right)(1-B)^d y_t = \left(1-\sum_{j=1}^{q}\theta_j B^j\right)\varepsilon_t.$$

Uncertainty Characterization via Scenarios

- One special kind of non-stationarity appears when the processes present a periodic or seasonal pattern.

✓ For instance, the daily seasonality indicates that a seasonal pattern of order equal to 24 can be identified in the series of hourly pool prices.

✓ This fact means that the price at hour hin day d is similar to the price at hour h in day d-1. Equivalent reasoning can be applied to explain the weekly seasonality, where the seasonality order is equal to $24 \times 7 = 168$.

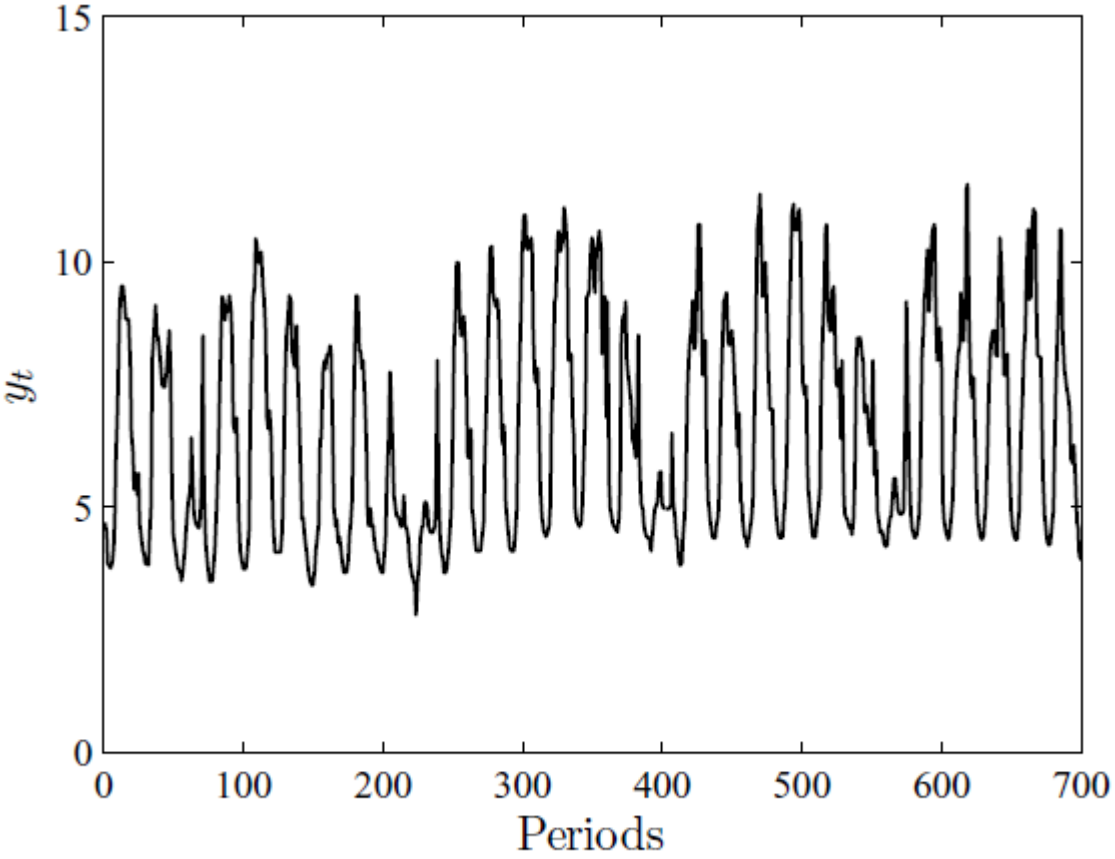


Fig. 3.5 Example of seasonal series.

Uncertainty Characterization via Scenarios

In this context, seasonal ARIMA models, also known as SARIMA, are required. Let us consider a stochastic process with a seasonality of order S. The general expression of a seasonal ARIMA model with parameters $(p, d, q) \times (P, D, Q)_S$ is

$$\left(1 - \sum_{j=1}^{p} \phi_j B^j\right) \left(1 - \sum_{j=1}^{P} \Phi_j B^{jS}\right) (1 - B)^d (1 - B^S)^D y_t = \left(1 - \sum_{j=1}^{q} \theta_j B^j\right) \left(1 - \sum_{j=1}^{Q} \Theta_j B^{jS}\right) \varepsilon_t$$

with a seasonal component of P autoregressive parameters Φ1, Φ2, . . ., ΦP , Q moving average parameters Θ1, Θ2, . . ., ΘQ, and a differentiation order D.

Uncertainty Characterization via Scenarios

Questioning the normality assumption. Normal Transformation

y_t boils down to a linear combination of white noises, and as such, the marginal distribution associated with the stochastic process $\boldsymbol{Y}$ is necessarily Gaussian, which is in accordance with the normality assumption on which ARMA processes are based.

In order to preserve the original marginal distribution associated with the stochastic process Y while making use of the modeling capability of the ARMA models, a new stochastic process, Z, with a standard normal marginal distribution is defined through the transformation:

$$\boldsymbol{Z} = \Phi^{-1}\left[F_Y(\boldsymbol{Y})\right]$$

where F_Y is the cumulative distribution function (cdf) of the marginal distribution associated with the original stochastic process $\boldsymbol{Y}$ and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal random variable.

Uncertainty Characterization via Scenarios

The ARMA model is adjusted to the transformed stochastic process Z. From the fitted model, a transformed set of scenarios Z is first generated, and then untransformed in order to produce a set of scenarios, Y , characterizing the original stochastic process, Y . The inverse transformation can be written as

$$\boldsymbol{Y} = F_Y^{-1}\left[\Phi(\boldsymbol{Z})\right]$$

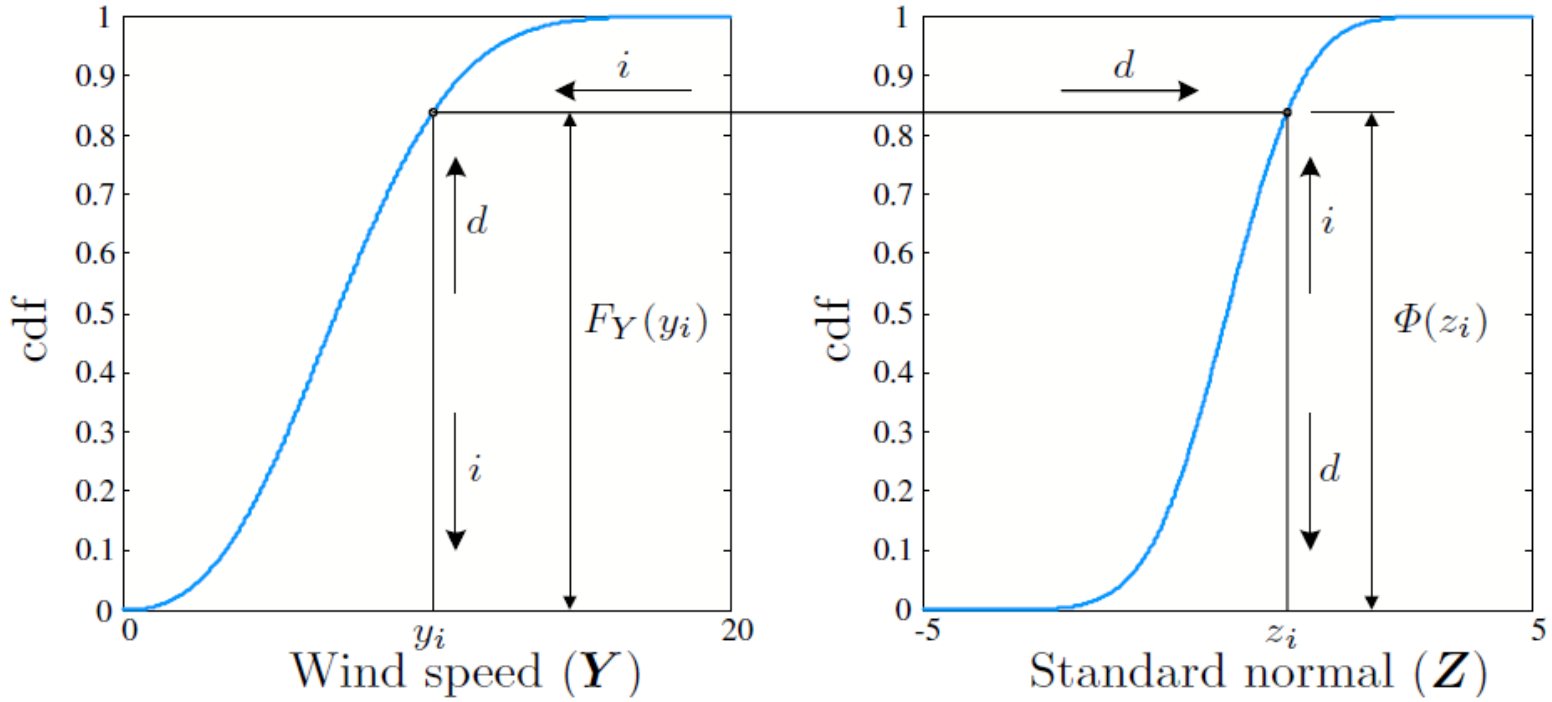


Fig. 3.6 Graphic interpretation of the normal transformation in equation (3.5).

Uncertainty Characterization via Scenarios

Example (Normal transformation)

Suppose that y1 =1.32 is a sample from an uniformly distributed random variable Y defined within the interval [1, 2]. Consequently, its cumulative distribution function (cdf), $F_Y(y)$, is given by

$$F_Y(y) = \begin{cases} 0, & \text{for } y < 1; \\ y - 1, & \text{for } 1 \leq y < 2; \\ 1, & \text{for } y \geq 2. \end{cases}$$

The direct transformation (3.5) is performed as follows:

$$F_Y(y_1) = 1.32 - 1 = 0.32 \rightarrow z_1 = \Phi^{-1}(0.32) = -0.47,$$

where z1 = −0.47 is a sample from a standard normal variable Z. Next, consider a new sample z2 = 0.7 from such a random variable Z. The inverse transformation is carried out as:

$$\Phi(z_2) = 0.76 \rightarrow y_2 = F_Y^{-1}(0.76) = 1.76,$$

where y2 = 1.76 is a sample from the uniformly distributed variable Y .

Uncertainty Characterization via Scenarios

❑ Generating Scenarios for Unit Availability

One way of reducing the financial risk faced by a producer that participates in the pool is using forward contracts. If the unit is not available when the contract is exercised, the producer is forced to buy energy in the pool in order to provide the energy previously committed in the forward contract.

- If the pool prices are smaller than the price associated with the forward contract, the producer will attain a profit.
- However, if the pool prices are greater than the forward contract price, the producer will incur financial losses.

Uncertainty Characterization via Scenarios

The availability of a unit is usually characterized by the time to failure (t_{F}) and the time to repair (t_{R}) which constitute random variables following exponential distributions, namely

$$t_{\mathrm{F}} = -\mathrm{MTTF} \times \ln(u_1)$$
$$t_{\mathrm{R}} = -\mathrm{MTTR} \times \ln(u_2),$$

where MTTF and MTTR are the mean time to failure and the mean time to repair, respectively, and u1 and u2 are random variables that are uniformly distributed between 0 and 1.

Uncertainty Characterization via Scenarios

Example (Time to failure and time to repair distribution functions)

- ✓ In this example we consider two generating units with parameters MTTF and MTTR equal to 100 and 10 hours for unit 1, and 150 and 20 hours for unit 2, respectively.

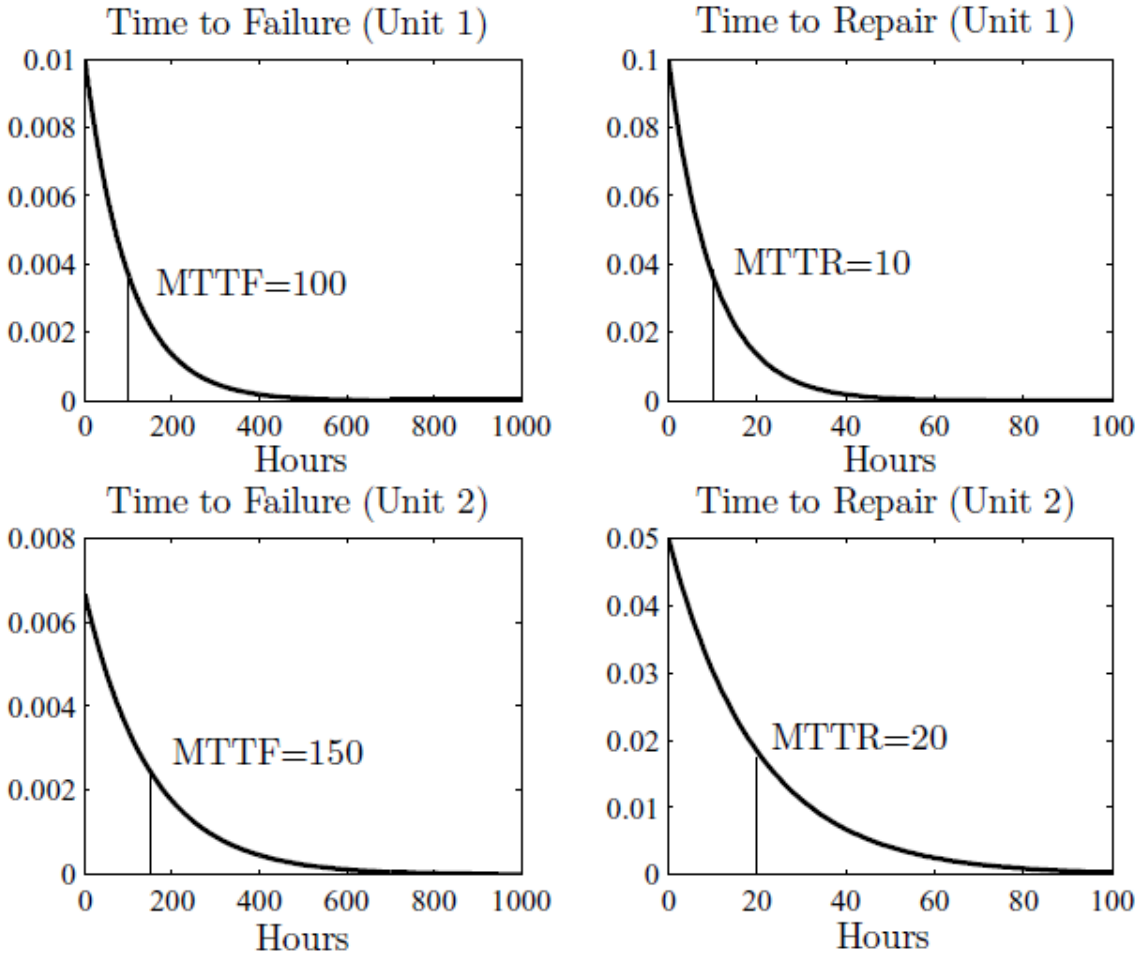


Fig. 3.7 Examples of time to failure and time to repair distributions.

Uncertainty Characterization via Scenarios

Observe that the expected values of the times to failure and to repair coincide with parameters MTTF and MTTR, respectively.

For a given generating unit, an availability scenario is a vector of 1s and 0s whose length is equal to the number of considered periods. If the t-element of the availability scenario is equal to 1, the unit is then available in period t, being unavailable if that element is equal to 0. The availability of unit g in period t and scenario ω is denoted by $a_{gt\omega}$.

Uncertainty Characterization via Scenarios

❑ Algorithm

- *Step 1*: Initialize the scenario counter: $\omega \leftarrow 0$.
- *Step 2*: Update the scenario counter and initialize the time period counter: $\omega \leftarrow \omega + 1$, $t \leftarrow 0$.
- *Step 3*: Randomly generate $u_1 \sim \mathrm{U}(0,1)$ and $u_2 \sim \mathrm{U}(0,1)$.
- *Step 4*: Evaluate expressions (3.7) and (3.8) to obtain t_{F} and t_{R}.
- *Step 5*: From $\min(N_{\mathrm{T}}, t+1)$ to $\min(N_{\mathrm{T}}, \mathrm{round}(t+t_{\mathrm{F}}))$, $a_{gt\omega} = 1$.
- *Step 6*: From $\min(N_{\mathrm{T}}, \mathrm{round}(t+t_{\mathrm{F}}+1))$ to $\min(N_{\mathrm{T}}, \mathrm{round}(t+t_{\mathrm{F}}+t_{\mathrm{R}}))$, $a_{gt\omega} = 0$.
- *Step 7*: Update the time period counter: $t \leftarrow \min(round(t+t_{\mathrm{F}}+t_{\mathrm{R}}+1),\ N_{\mathrm{T}})$.
- *Step 8*: If $t < N_{\mathrm{T}}$ go to Step 3), else go to Step 9)
- *Step 9*: If $\omega < N_{\Omega}$ go to Step 2), else the scenario-generation process concludes.

Uncertainty Characterization via Scenarios

Example (Availability scenarios)

Suppose a planning horizon of 2000 hours, and a generating unit with MTTF and MTTR equal to 1000 and 50 hours, respectively. The first step consists in randomly generating values from uniform random variables u1 and u2 in order to obtain the time to failure and the time to repair, respectively.

- In this case, we obtain u1 = 0.6324 and u2 = 0.0975, resulting in times to failure and to repair equal to 458.3 and 116.4 hours, respectively. Since we are using a discrete number of periods (2000 hours), we need to round those real numbers to integer values, resulting in 458 and 116 hours, respectively. This way, during the first 458 hours the unit is available. In hour 459 the unit breaks down and remains unavailable during 116 hours.
- The procedure continues by generating new values for u1 and u2 until the planning horizon is covered. Thus, we generate u1 = 0.2785 and u2 = 0.5469, which translates into times to failure and repair equal to 1278 and 30 hours, in that order. At this point, 458 + 116 + 1278 + 30 = 1882 hours of the planning horizon have been completed, while 2000 – 1882 = 118 hours are still uncovered.

Uncertainty Characterization via Scenarios

- Next, we obtain u1 = 0.8003 and u2 = 0.1419, which yield times to failure and repair equal to 223 and 98, respectively. Observe that the time to failure, 223, is greater than the number of remaining hours, 118. Consequently, the unit remains available until the end of the planning horizon.

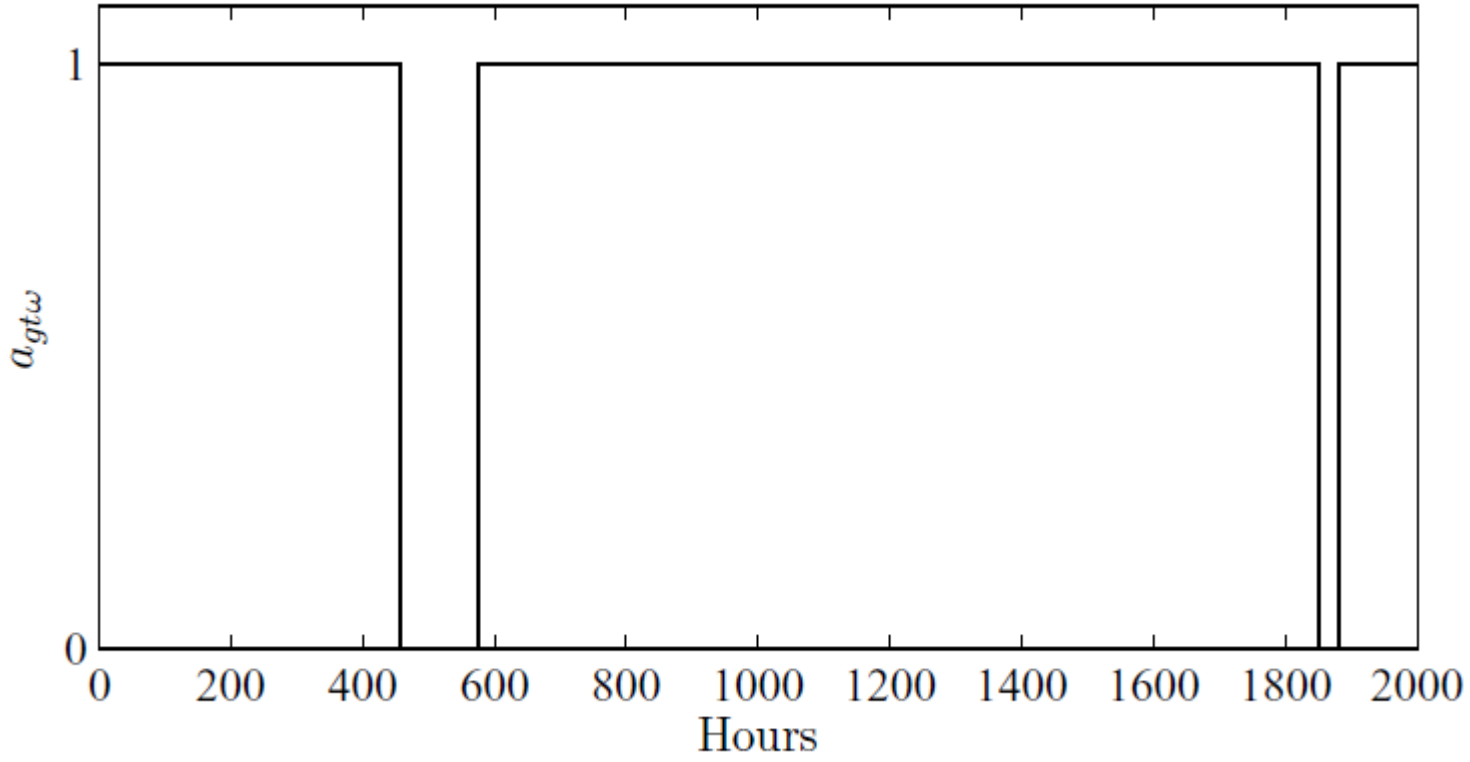


Fig. 3.8 Example of an availability scenario.

Uncertainty Characterization via Scenarios

- Significant effort has been made in order to measure the error associated with the usage of a discrete representation of the stochastic process instead of the continuous one.
- This error is usually expressed in terms of the difference between the objective functions of the discrete and continuous stochastic problems.
- The first requirement is the stability of the scenario trees. In other words, several trees randomly obtained using the same scenario generation procedure should provide (almost) the same solution to the stochastic problem considered.

✓ If these solutions are (almost) the same, we can say that there exists in-sample stability.
✓ If, in addition, we evaluate the true objective function for the set of optimal decisions that results from each scenario tree and we obtain (almost) the same values, then we are in an out-of-sample stability case.
✓ Clearly, since in this case it is necessary to compute the true objective function, testing out-of-sample stability is more complicated than testing in-sample stability.

Uncertainty Characterization via Scenarios

- The second requirement is that the solution obtained with the scenario tree should be unbiased with respect to the true solution obtained with the continuous process.
- This requirement is difficult to check because, as we already know, the true solution is unknown. For this reason, an approximate method consists in building a reference tree representing the continuous (or true) process as well as possible.
- The optimal decisions obtained from the trial tree can be tested on the reference tree and the resulting value of the objective function can be compared with that obtained from solving the problem with the reference tree directly.
- The reference tree should be as large as possible and it should not be generated using the same procedure than that used for generating the trial tree.
- Another relevant issue for discussion is the shape of the scenario tree. That is, it must be determined how many stages the tree comprises and how many branches leave each node.

Uncertainty Characterization via Scenarios

❑ Scenario Reduction

- The scenario-generation technique described is based on a sampling approach.
- Consequently, in order for this approximation to be accurate, a high number of scenarios is usually required.
- Given that the computational burden of a stochastic programming model rapidly increases with the number of scenarios, a mathematical tool aimed to shrewdly reduce such a number becomes a need.
- In other words, the necessity of reconciling scenario generation and computational tractability is what justifies the existence of scenario-reduction techniques.
- In more detail, a scenario-reduction methodology seeks to downsize a scenario set while still keeping as intact as possible the stochastic information embedded in it.

Uncertainty Characterization via Scenarios

❑ Scenario Reduction Using a Probability Distance

- In two-stage stochastic linear programming problems, it is possible to reduce a large scenario set to a simpler one that is close to the original one if measured by a so-called probability distance.

- Under mild conditions on the problem data, it can be shown that the optimal value of the simpler problem (the one involving the reduced scenario set) is close to the value of the solution to the original problem (the one considering the original scenario set) if the scenario sets are sufficiently close in terms of such a probability distance.

Uncertainty Characterization via Scenarios

The most common probability distance used in stochastic programming is the *Kantorovich distance*, $D_K(\cdot)$, defined between two discrete probability distributions Q and Q' by the following problem known as *Monge-Kantorovich mass transportation* problem.

$$D_K(Q,Q') = \min_{\eta}\Bigg\{ \sum_{\substack{\omega\in\Omega \\ \omega'\in\Omega_S}} \nu(\omega,\omega')\eta(\omega,\omega') :$$

$$\eta(\omega,\omega') \geq 0,\ \forall\omega\in\Omega, \forall\omega'\in\Omega_S,$$

$$\sum_{\omega'\in\Omega_S} \eta(\omega,\omega') = \pi_\omega,\ \forall\omega\in\Omega,$$

$$\sum_{\omega\in\Omega} \eta(\omega,\omega') = \tau_{\omega'},\ \forall\omega'\in\Omega_S \Bigg\},$$

where π_ω and $\tau_{\omega'}$ represent the probabilities of scenarios ω and ω' in sets and S according to probability distributions Q and Q' , respectively.

Uncertainty Characterization via Scenarios

- ✓ Given that the electricity-market problems addressed in this book solely present stochasticity in the objective function and right-hand sides, the Kantorovich distance can be equivalently determined as

$$D_K(Q, Q') = \sum_{\omega \in \Omega \setminus \Omega_S} \pi_\omega \min_{\omega' \in \Omega_S} \nu(\omega, \omega').$$

- ✓ Specifically, two different heuristic algorithms can be developed, namely, the backward reduction or the forward selection, depending on whether the subset $\Omega_S \subset \Omega$ is built by eliminating or selecting scenarios from the initial set Ω.

- ✓ Electricity-market problems tackled via stochastic programming are usually computationally intensive, and as such, the scenario sets representing the stochastic processes involved often require a preliminary drastic reduction so as to enable tractability and/or enhance efficiency. In these cases, the *forward selection* algorithm exhibits a better performance. For this reason, we focus our study on this algorithm hereinafter.

Uncertainty Characterization via Scenarios

❑ Forward selection Algorithm

Let us denote the original scenario set as Ω. The forward selection algorithm produces a succession of reduced sets $\Omega_S^{[0]}, \Omega_S^{[1]}, \ldots, \Omega_S^{[i]}, \ldots, \Omega_S^*$, of increasing cardinality, where the set Ω_S^* is the target of the search. To this end, the forward selection algorithm works as follows:

- *Step 0*: The initial step consists in computing the function $\nu(\cdot)$ defining the Kantorovich distance for each pair of scenarios. Two different variants of this algorithm are defined depending on how the cost function $\nu(\cdot)$ is calculated. These variants are explained below in detail.

- *Step 1*: The iterative process begins with the choice of a starting scenario, that is, with the scenario from which the reduced scenario set is built. Mathematically, the starting scenario (ω_1) is obtained from

Uncertainty Characterization via Scenarios

$$\omega_1 = \arg\{\min_{\omega' \in \Omega} \sum_{\omega \in \Omega} \pi_\omega \nu(\omega, \omega')\}.$$

- *Step* i: Starting from the scenario selected in step 1, in each iteration a new scenario is added to the reduced scenario set until it is considered to be close enough to the original one. For this purpose, this selection is carried out using

$$\omega_i = \arg\left\{\min_{\omega' \in \Omega_J^{[i-1]}} \sum_{\omega \in \Omega_J^{[i-1]} \setminus \{\omega'\}} \pi_\omega \min_{\omega'' \in \Omega_S^{[i-1]} \bigcup \{\omega\}} \nu(\omega, \omega'')\right\},$$

Uncertainty Characterization via Scenarios

- *Step* $N_{\Omega_S^*}+1$: In this step, an optimal redistribution of probabilities is carried out. It consists of adding the probabilities of those scenarios which have not been finally selected $\left(\omega \in \Omega_J^*, \text{ with } \Omega_J^* \cup \Omega_S^* = \Omega\right)$ to the probabilities of those comprising the reduced set $(\omega \in \Omega_S^*)$.
  Mathematically, this redistribution of probabilities can be accomplished as follows,

$$\pi_\omega^* = \pi_\omega + \sum_{\omega' \in J(\omega)} \pi_{\omega'},$$

  where $J(\omega)$ is defined as the set of scenarios $\omega' \in \Omega_J^*$ such that $\omega = \arg\min_{\omega'' \in \Omega_S^*} \nu(\omega'', \omega')$.

✓ A particularly efficient implementation of the algorithm described above is presented next and is referred to as *fast forward selection algorithm*.

Uncertainty Characterization via Scenarios

- *Step 0*: Compute cost function $\nu(\omega, \omega')$ for each pair of scenarios ω and ω' in Ω.
- *Step 1*: Compute $d_\omega = \sum_{\substack{\omega'=1 \\ \omega' \neq \omega}}^{N_\Omega} \pi_{\omega'} \nu(\omega, \omega'), \quad \forall \omega \in \Omega.$

  Select $\omega_1 \in \arg \min_{\omega \in \Omega} d_\omega$.

  Set $\Omega_J^{[1]} = \{1, \ldots, N_\Omega\} \setminus \omega_1$.
- *Step i*: Compute

  $$\nu^{[i]}(\omega, \omega') = \min\{\nu^{[i-1]}(\omega, \omega'), \nu^{[i-1]}(\omega, \omega_{i-1})\}, \forall \omega, \omega' \in \Omega_J^{[i-1]}$$
  $$d_\omega^{[i]} = \sum_{\omega' \in \Omega_J^{[i-1]} \setminus \{\omega\}} \pi_{\omega'} \nu^{[i]}(\omega', \omega), \quad \forall \omega \in \Omega_J^{[i-1]}.$$

  Select $\omega_i \in \arg \min_{\omega \in \Omega_J^{[i-1]}} d_\omega^{[i]}$.

  Set $\Omega_J^{[i]} = \Omega_J^{[i-1]} \setminus \omega_i$.
- *Step* $N_{\Omega_S^*} + 1$:

  $\Omega_J^* = \Omega_J^{[N_{\Omega_J^*}]}$ is the final set of indices of discarded scenarios.

  $\Omega_S^* = \Omega \setminus \Omega_J^*$ is the set of indices of selected scenarios.

  The probabilities of selected scenarios $\omega \in \Omega_S^*$ are computed as

  $\pi_\omega^* \leftarrow \pi_\omega + \sum_{\omega' \in J(\omega)} \pi_{\omega'}, \forall \omega \in \Omega_S^*$, where

  $J(\omega) = \{\omega' \in \Omega_J^* | \omega = j(\omega')\}, \quad j(\omega') \in \arg \min_{\omega'' \in \Omega_S^*} \nu(\omega'', \omega').$

Uncertainty Characterization via Scenarios

❑ Variant 1: Cost function based on the norm of the difference between pairs of random vectors

Consider a stochastic process $\boldsymbol{\lambda}$ characterized by an initial set of scenarios $\boldsymbol{\lambda}(\omega)$, for all $\omega \in \Omega$ i.e., $\boldsymbol{\lambda} = \{\boldsymbol{\lambda}(\omega), \omega = 1, 2, \ldots, N_\Omega\}$

Cost function $\nu(\cdot)$ can be computed as the norm of the difference between pairs of scenarios, i.e.,

$$\nu(\omega, \omega') = \|\boldsymbol{\lambda}(\omega) - \boldsymbol{\lambda}(\omega')\|,$$

where $\|\cdot\|$ stands for the norm operator.

✓ This first variant of the forward selection algorithm yields the same reduced set Ω_S irrespective of the stochastic programming problem to be solved.

Uncertainty Characterization via Scenarios

❑ Variant 2: Cost function based on the objective function of a single-scenario problem

Consider the deterministic equivalent formulation of a two-stage stochastic programming problem

$$\text{Minimize}_{\boldsymbol{x},\boldsymbol{y}(\omega)}$$

$$z = \boldsymbol{c}^{\top}\boldsymbol{x} + \sum_{\omega\in\Omega} \pi(\omega)\boldsymbol{q}(\omega)^{\top}\boldsymbol{y}(\omega)$$

$$\text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$

$$\boldsymbol{T}(\omega)\boldsymbol{x} + \boldsymbol{W}(\omega)\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega), \forall\omega \in \Omega$$

$$\boldsymbol{x} \in X,\ \boldsymbol{y}(\omega) \in Y;\ \forall\omega \in \Omega,$$

Define $\mathrm{DP}_{\mathcal{E}}$ as the deterministic expected value problem associated with SP, i.e., $\mathrm{DP}_{\mathcal{E}}$ is equal to problem SP in which the random vector $\boldsymbol{\lambda}$ is replaced by its expected value, $\bar{\boldsymbol{\lambda}} = \mathcal{E}\{\boldsymbol{\lambda}\}$. Mathematically, this problem can be formulated as followings.

Uncertainty Characterization via Scenarios

$$\text{Minimize}_{\boldsymbol{x},\boldsymbol{y}(\bar{\boldsymbol{\lambda}})}$$
$$z_{\mathcal{E}} = \boldsymbol{c}^{\top}\boldsymbol{x} + \boldsymbol{q}(\bar{\boldsymbol{\lambda}})^{\top}\boldsymbol{y}(\bar{\boldsymbol{\lambda}})$$
$$\text{subject to}$$
$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$
$$\boldsymbol{T}(\bar{\boldsymbol{\lambda}})\boldsymbol{x} + \boldsymbol{W}(\bar{\boldsymbol{\lambda}})\boldsymbol{y}(\bar{\boldsymbol{\lambda}}) = \boldsymbol{h}(\bar{\boldsymbol{\lambda}})$$
$$\boldsymbol{x} \in X,\ \boldsymbol{y}(\bar{\boldsymbol{\lambda}}) \in Y.$$

Let us denote the optimal value of the first-stage variables in problem $\mathrm{DP}_{\mathcal{E}}$ as $\bar{\boldsymbol{x}}$.

Next, we build the single-scenario problem DP_{ω} as the one resulting from problem SP if the stochastic process $\boldsymbol{\lambda}$ is replaced by its realization in scenario ω, $\boldsymbol{\lambda}(\omega)$, and its first-stage variables are fixed to $\bar{\boldsymbol{x}}$, i.e.,

$$\text{Minimize}_{\boldsymbol{y}(\omega)}$$
$$z_{\omega} = \boldsymbol{c}^{\top}\bar{\boldsymbol{x}} + \boldsymbol{q}(\omega)^{\top}\boldsymbol{y}(\omega)$$
$$\text{subject to}$$
$$\boldsymbol{T}(\omega)\bar{\boldsymbol{x}} + \boldsymbol{W}(\omega)\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega)$$
$$\boldsymbol{y}(\omega) \in Y.$$

Uncertainty Characterization via Scenarios

Finally, cost function $\nu(\cdot)$ is defined as

$$\nu(\omega, \omega') = |z_\omega - z_{\omega'}|,$$

where z_ω is the optimal value of the objective function of problem DP_ω.

- Unlike in variant 1, the reduction process in this second variant of the forward selection algorithm is dependent on the stochastic programming problem SP to be solved to obtair z_ω .
- In other words, this variant incorporates information on the structure of the problem into the cost function v(·), which, in general, results in reduced sets that yield better solutions in practice.
- Consequently, this second variant is computationally more demanding than the first one.

Uncertainty Characterization via Scenarios

Example (Fast forward selection: Variant 1)

Assume that the power production P of a 100-MW wind farm in a given time period can be statistically represented by the four wind power scenarios $P_\omega, \omega = 1, \ldots, 4,$, listed in Table below, with associated probabilities $\pi_\omega, \omega = 1, \ldots, 4.$

Illustrative Example 3.10: wind power scenarios

Scenario #	1	2	3	4
P_ω (MW)	5	40	60	100
π_ω	0.25	0.20	0.20	0.35

Suppose that we need to reduce the number of wind power scenarios to two, i.e., $N_{\Omega_S^*} = 2$.
For this purpose, we decide to use variant 1 of the fast forward selection algorithm, which works as follows:

Uncertainty Characterization via Scenarios

- Step 0 : Calculation of cost function as $\nu(\omega,\omega') = \|P_\omega - P_{\omega'}\|, \forall \omega,\omega' \in \Omega$, where $\Omega = \{1,2,3,4\}$.

$$\nu = \begin{pmatrix} 0 & 35 & 55 & 95 \\ 35 & 0 & 20 & 60 \\ 55 & 20 & 0 & 40 \\ 95 & 60 & 40 & 0 \end{pmatrix} \text{MW}.$$

- Step 1 : For each scenario ω candidate to be selected from Ω, we choose the one that minimizes the resulting Kantorovich distance between the reduced and original sets, Ω_S and Ω, respectively.

$$\begin{aligned} d_1 &= \pi_2\,\nu(1,2) + \pi_3\,\nu(1,3) + \pi_4\,\nu(1,4) = 51.25 \text{ MW} \\ d_2 &= \pi_1\,\nu(2,1) + \pi_3\,\nu(2,3) + \pi_4\,\nu(2,4) = 33.75 \text{ MW} \\ d_3 &= \pi_1\,\nu(3,1) + \pi_2\,\nu(3,2) + \pi_4\,\nu(3,4) = 31.75 \text{ MW} \\ d_4 &= \pi_1\,\nu(4,1) + \pi_2\,\nu(4,2) + \pi_3\,\nu(4,3) = 43.75 \text{ MW}. \end{aligned}$$

Therefore,

$$\begin{aligned} \Omega_S^{[1]} &= \{3\}, \\ \Omega_J^{[1]} &= \{1,2,4\}. \end{aligned}$$

Uncertainty Characterization via Scenarios

- Step 2 : Next, we update the cost matrix as follows:

$$\begin{aligned}
\nu^{[2]}(1,2) &= \min\{\nu(1,2),\nu(1,3)\} = 35 \text{ MW}\\
\nu^{[2]}(1,4) &= \min\{\nu(1,4),\nu(1,3)\} = 55 \text{ MW}\\
\nu^{[2]}(2,1) &= \min\{\nu(2,1),\nu(2,3)\} = 20 \text{ MW}\\
\nu^{[2]}(2,4) &= \min\{\nu(2,4),\nu(2,3)\} = 20 \text{ MW}\\
\nu^{[2]}(4,1) &= \min\{\nu(4,1),\nu(4,3)\} = 40 \text{ MW}\\
\nu^{[2]}(4,2) &= \min\{\nu(4,2),\nu(4,3)\} = 40 \text{ MW}.
\end{aligned}$$

Consequently,

$$\nu^{[2]} = \begin{pmatrix} 0 & 35 & 55 & 55\\ 20 & 0 & 20 & 20\\ 55 & 20 & 0 & 40\\ 40 & 40 & 40 & 0 \end{pmatrix} \text{MW}.$$

Considering the new cost matrix , we select from $\Omega_I^{[1]}$ the scenario ω that, if introduced into the subsequent reduced $\Omega_S^{[2]}$, minimizes the Kantorovich distance between this set and the original one, Ω ,

$$\begin{aligned}
d_1^{[2]} &= \pi_2\ \nu^{[2]}(2,1) + \pi_4\ \nu^{[2]}(4,1) = 18.00 \text{ MW}\\
d_2^{[2]} &= \pi_1\ \nu^{[2]}(1,2) + \pi_4\ \nu^{[2]}(4,2) = 22.75 \text{ MW}\\
d_4^{[2]} &= \pi_1\ \nu^{[2]}(1,4) + \pi_2\ \nu^{[2]}(2,4) = 17.75 \text{ MW}.
\end{aligned}$$

Uncertainty Characterization via Scenarios

Hence,

$$\Omega_S^{[2]} = \Omega_S^* = \{3, 4\}$$
$$\Omega_J^{[2]} = \Omega_J^* = \{1, 2\}.$$

- Step 3 : The scenario-reduction algorithm ends with the optimal transfer of probabilities from the set of non-selected scenarios.

1. Scenario 3 in Ω_S^* is the closest one to scenario 1 in Ω_J^* $(\nu(1,3) = 55$, while $\nu(1,4) = 95)$, and
2. Scenario 3 in Ω_S^* is the closest one to scenario 2 in Ω_J^* $(\nu(2,3) = 20$, while $\nu(2,4) = 60)$,

It follows

$$\pi_3^* = \pi_3 + \pi_2 + \pi_1 = 0.65$$
$$\pi_4^* = \pi_4 = 0.35.$$

Uncertainty Characterization via Scenarios

Example (Fast forward selection: Variant 2)

Previous to tackling the reduction process itself, this variant requires to define the problem to be solved considering this scenario set (problem SP).

Let us assume that the producer owning the wind farm is interested in selling its wind power production to the day-ahead electricity market. Since the final power produced by its wind farm is uncertain, the producer has no option other than to submit an energy offer (P^D) to the day-ahead market, subsequently fixing its likely excess (Δ^+) or deficit of generation (Δ^-) through a real-time balancing process.

Uncertainty Characterization via Scenarios

The decision on the amount of energy to be sold in the day-ahead market with a view to minimize the imbalance cost can be properly made by means of the following optimization problem,

$$\text{Maximize}_{P^{\mathrm{D}};\Delta_{\omega}^{+},\forall\omega;\Delta_{\omega}^{-},\forall\omega}$$

$$z=\sum_{\omega=1}^{N_{\Omega}}\pi_{\omega}\left(\lambda^{\mathrm{D}}P_{\omega}-\lambda^{\mathrm{D}}(1-r^{+})\Delta_{\omega}^{+}-\lambda^{\mathrm{D}}(r^{-}-1)\Delta_{\omega}^{-}\right)$$

$$\text{subject to}$$

$$0\leq P^{\mathrm{D}}\leq P^{\max}$$

$$\Delta_{\omega}^{+}-\Delta_{\omega}^{-}=P_{\omega}-P^{\mathrm{D}},\forall\omega$$

$$0\leq\Delta_{\omega}^{+}\leq P_{\omega},\forall\omega$$

$$0\leq\Delta_{\omega}^{-}\leq P^{\max},\forall\omega,$$

- Step 0 : Calculate cost function $\nu(\omega,\omega')=|z_{\omega}-z_{\omega'}|,\forall\omega,\omega'\in\Omega.$

Uncertainty Characterization via Scenarios

we need to solve first the following deterministic expected value problem $\mathrm{DP}_{\mathcal{E}}$,

$$\begin{aligned}
&\text{Maximize}_{\bar{P}^{\mathrm{D}},\bar{\Delta}^{+},\bar{\Delta}^{-}} \\
&z_{\mathcal{E}} = (1125 - \bar{\Delta}^{+} - 8\bar{\Delta}^{-}) \\
&\text{subject to} \\
&0 \le \bar{P}^{\mathrm{D}} \le 100 \\
&\bar{\Delta}^{+} - \bar{\Delta}^{-} = 56.25 - \bar{P}^{\mathrm{D}} \\
&0 \le \bar{\Delta}^{+} \le 56.25 \\
&0 \le \bar{\Delta}^{-} \le 100,
\end{aligned}$$

$$\mathcal{E}\{P\} = 5 \times 0.25 + 40 \times 0.20 + 60 \times 0.20 + 100 \times 0.35 = 56.25 \text{ MW}.$$

✓ The solution to this problem is trivial: $\bar{P}^{\mathrm{D}} = 56.25$ MW, $\bar{\Delta}^{+} = \bar{\Delta}^{-} = 0$, and $z_{\mathcal{E}} = \$1125$.

Uncertainty Characterization via Scenarios

Finally, the objective function values $z_\omega, \forall \omega \in \Omega,$, are obtained from the family of single-scenario problems $\mathrm{DP}_\omega, \omega = 1, \ldots, 4,$, which result from problem SP if the wind power stochastic process is replaced by its realization in scenario ω, P_ω , and its first-stage variable P^{D} is fixed to $\bar{P}^{\mathrm{D}} = 56.25$ MW. Specifically, each problem DP_ω is formulated as follows:

$$\mathrm{Maximize}_{\Delta_\omega^+;\Delta_\omega^-}$$

$$z_\omega = (20P_\omega - \Delta_\omega^+ - 8\Delta_\omega^-)$$

subject to

$$\Delta_\omega^+ - \Delta_\omega^- = P_\omega - 56.25$$

$$0 \leq \Delta_\omega^+ \leq P_\omega$$

$$0 \leq \Delta_\omega^- \leq 100.$$

Uncertainty Characterization via Scenarios

The solution to this problem is also straightforward and is given by

$$\Delta_{\omega}^{+}\ (\text{MW}) = \begin{cases} 0, & \text{if } \omega = 1, 2 \\ P_{\omega} - 56.25, & \text{if } \omega = 3, 4. \end{cases}$$

$$\Delta_{\omega}^{-}\ (\text{MW}) = \begin{cases} 56.25 - P_{\omega}, & \text{if } \omega = 1, 2 \\ 0, & \text{if } \omega = 3, 4. \end{cases}$$

$$z_{\omega}\ (\$) = \begin{cases} 28P_{\omega} - 450, & \text{if } \omega = 1, 2 \\ 19P_{\omega} + 56.25, & \text{if } \omega = 3, 4. \end{cases}$$

Scenario #	1	2	3	4
z_{ω} ($)	-310	670	1196.25	1956.25
π_{ω}	0.25	0.20	0.20	0.35

Uncertainty Characterization via Scenarios

Therefore, we already have the required information to compute cost matrix ν, which results in

$$\nu = \$ \begin{pmatrix} 0 & 980 & 1506.25 & 2266.25 \\ 980 & 0 & 526.25 & 1286.25 \\ 1506.25 & 526.25 & 0 & 760 \\ 2266.25 & 1286.25 & 760 & 0 \end{pmatrix}.$$

From this point on, this second variant of the fast forward selection algorithm proceeds similarly to variant 1 in previous example.

Final results:

$$\Omega_S^{[2]} = \Omega_S^* = \{1, 3\}$$
$$\Omega_J^{[2]} = \Omega_J^* = \{2, 4\}.$$
$$\pi_1^* = \pi_1 = 0.25$$
$$\pi_3^* = \pi_3 + \pi_2 + \pi_4 = 0.75.$$

Uncertainty Characterization via Scenarios

❑ Scenario Generation for Dependent Stochastic Processes

- Decision-making problems related to electricity markets may be affected by stochastic processes that are statistically dependent.
- It is well-known that electricity prices and demand are strongly interrelated in such a way that periods of high/low prices often coincide with periods of high/low demand.
- In short, dependencies among the stochastic processes modeling uncertainty in optimization problems may have a significant impact on decision variables, and therefore, procedures capable of generating scenario sets incorporating such dependencies are required.

Uncertainty Characterization via Scenarios

Let us consider two stochastic processes, $\boldsymbol{Y}^a$ and $\boldsymbol{Y}^b$, that can be separately described by means of the following ARMA (*p*,*q*) models:

$$y_t^a = \sum_{j=1}^{p^a} \phi_j^a y_{t-j}^a + \varepsilon_t^a - \sum_{j=1}^{q^a} \theta_j^a \varepsilon_{t-j}^a$$

$$y_t^b = \sum_{j=1}^{p^b} \phi_j^b y_{t-j}^b + \varepsilon_t^b - \sum_{j=1}^{q^b} \theta_j^b \varepsilon_{t-j}^b \ .$$

Consequently, ε^a and ε^b should be indeed cross-correlated, i.e., $\mathcal{E}\{\varepsilon_t^a \varepsilon_{t-j}^b\} \neq 0$ and/or $\mathcal{E}\{\varepsilon_t^b \varepsilon_{t-j}^a\} \neq 0$ for some j. The dependency structure of the residual series ε^a and ε^b can be analyzed by means of their cross-correlogram, in which the cross-correlation coefficients for different time lags are represented.

Uncertainty Characterization via Scenarios

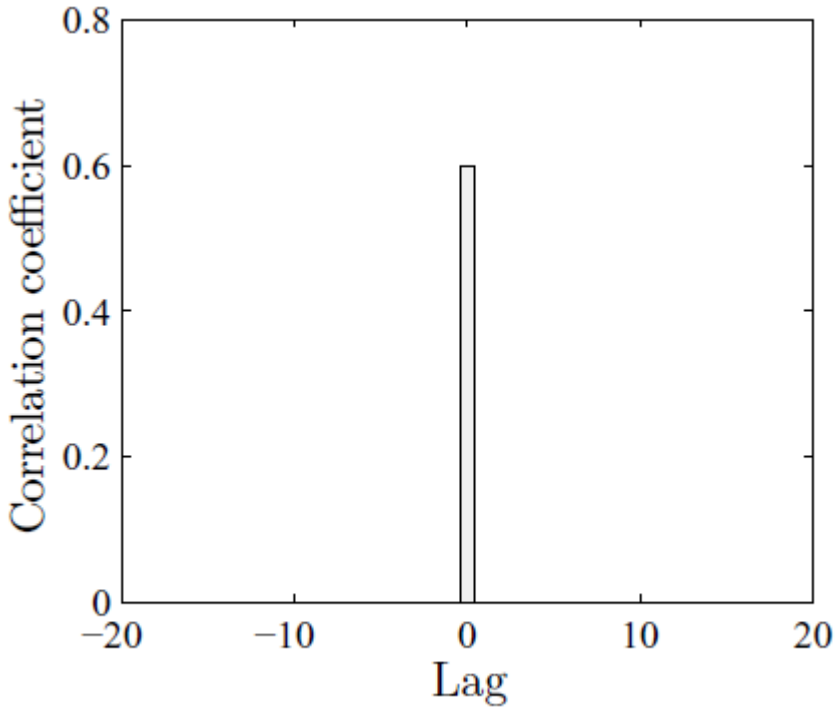
Specifically, the positive and negative lag-k cross-correlation coefficients, ρ_k and ρ_{-k}, of ε^a and ε^b are defined, respectively, as $\mathcal{E}\{\varepsilon_t^a \varepsilon_{t-k}^b\} = \rho_k$ and $\mathcal{E}\{\varepsilon_{t-k}^a \varepsilon_t^b\} = \rho_{-k}$, where the equality holds for any t by appealing to the stationarity assumption. The cross-correlation coefficient corresponding to the lag zero, ρo, is usually referred to as *contemporaneous correlation*.

❑ Contemporaneous stochastic processes.

The residual cross-correlogram of two contemporaneous stochastic processes is characterized by a single significative correlation coefficient placed at lag zero, that is, by a significative contemporaneous correlation.

The rest of coefficients can be considered statistically zero.

Contemporaneous stochastic processes are usually driven by a common physical cause.

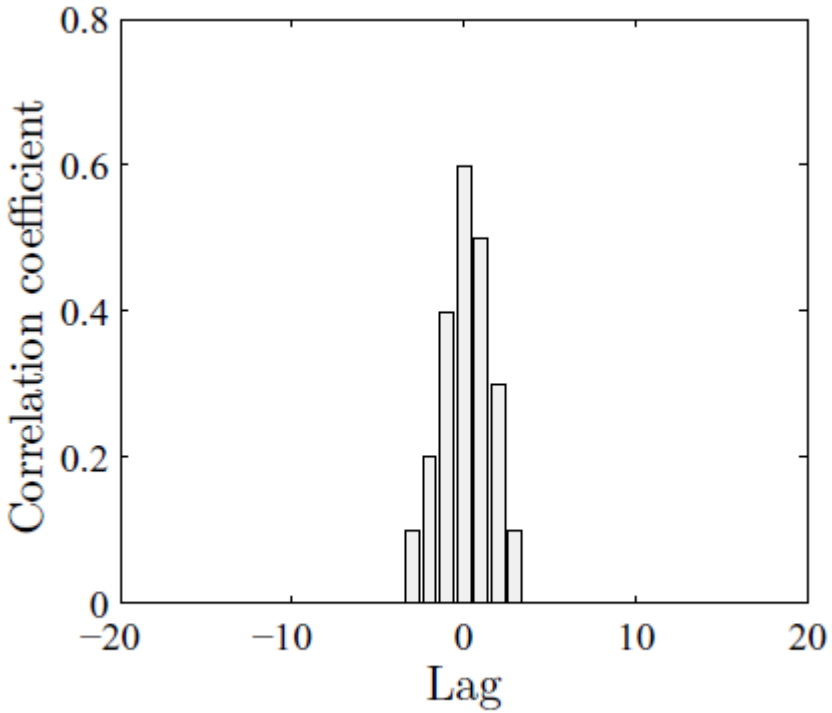


(a) Contemporaneous

Uncertainty Characterization via Scenarios

- **Quasi-contemporaneous stochastic processes.**

The residual cross-correlogram of two quasi-contemporaneous stochastic processes presents a triangular structure in which the most significant correlation coefficients form a group around the lag zero, with the lag-o correlation coefficient being precisely the largest one.

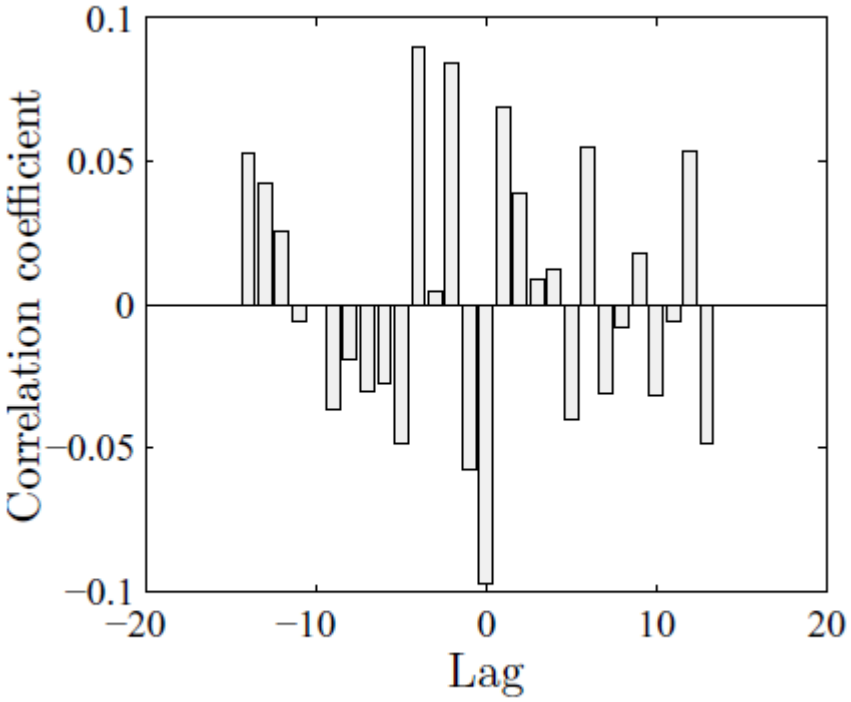


(b) Quasi-contemporaneous

Uncertainty Characterization via Scenarios

- **Non-contemporaneous stochastic processes.**

With this name, we refer to any other pair of correlated stochastic processes that does not fit into any of the two previous categories. The modeling of the dependencies existing between two non-contemporaneous stochastic processes is generally more complex and cumbersome.



(c) Non-contemporaneous

Uncertainty Characterization via Scenarios

❑ Scenarios for contemporaneous or quasi-contemporaneous stochastic processes

The methodology described below seeks to produce correlated scenarios by simulating a series of errors, ε^a and ε^b , that reproduces the residual crosscorrelogram of the stochastic processes, Y^a and Y^b, under consideration.

Mathematically, the cross-correlations between the series of errors ε^a and ε^b can be described by means of a variance-covariance matrix **G**.

$$\varepsilon = \begin{pmatrix} \varepsilon^a \\ \varepsilon^b \end{pmatrix} = B\xi = B \begin{pmatrix} \xi^a \\ \xi^b \end{pmatrix},$$

in which **B** is the orthogonal matrix of the transformation, and ξ is normal and satisfies that $E[\xi\xi^T] = I$, being I the identity matrix. That is, ξ is neither autocorrelated, nor cross-correlated. This way, white noises (independent standard normal errors) can be generated, and then, cross-correlated according to the variance-covariance matrix **G** by using the orthogonal transformation.

Uncertainty Characterization via Scenarios

In most engineering applications, matrix G, besides being symmetric, is also positive definite, and as such, can be decomposed through the computationally advantageous Cholesky decomposition, which avoids the calculation of eigenvalues and eigenvectors. The Cholesky decomposition can be stated as

$$G = LL^T,$$

where **L** is an inferior triangular matrix that turns out to be the orthogonal matrix required for transformation, i.e., **B** = **L**, as shown below.

$$\begin{aligned} G &= cov(\varepsilon, \varepsilon^T) = cov(B\xi, \xi^T B^T) = B\, cov(\xi, \xi^T) B^T \\ &= B\, E[\xi\xi^T] B^T = BIB^T = BB^T, \end{aligned}$$

$$G = LL^T = BB^T \Rightarrow (B^{-1}L)(L^T(B^T)^{-1}) = I \Rightarrow B = L.$$

Uncertainty Characterization via Scenarios

The procedure to generate a scenario set for a number N_S of quasi-contemporaneous stochastic processes is described step by step.

❑ Algorithm

- *Step 1*: For each stochastic process, fit an ARMA model to the corresponding series of historical values. This fitting process yields non-autocorrelated normal residuals (historical errors).
- *Step 2*: Although the historical error series obtained in the preceding step are not *autocorrelated*, they should be *cross-correlated* if the stochastic processes under analysis are indeed interrelated. In this step, the cross-correlations are computed and, consequently, the variance-covariance matrix $\boldsymbol{G}$ is built. Considering for the sake of simplicity just two stochastic processes, $\boldsymbol{Y}^a$ and $\boldsymbol{Y}^b$, the structure of matrix $\boldsymbol{G}$ is as

$$\boldsymbol{G} = \begin{array}{c} \begin{array}{cc} \varepsilon^a & \varepsilon^b \end{array} \\ \left(\begin{array}{c|c} \boldsymbol{G}_{11} & \boldsymbol{G}_{12} \\ \hline \boldsymbol{G}_{21} & \boldsymbol{G}_{22} \end{array}\right) \end{array} \begin{array}{l} \varepsilon^a \\ \varepsilon^b \end{array}$$

Uncertainty Characterization via Scenarios

where $\boldsymbol{G}_{11} = \sigma^2_{\varepsilon^a} \boldsymbol{I}_{K+N_T}$, $\boldsymbol{G}_{22} = \sigma^2_{\varepsilon^b} \boldsymbol{I}_{K+N_T}$, and

$$\boldsymbol{G}_{12} = \boldsymbol{G}_{21}^T = \sigma_{\varepsilon^a} \sigma_{\varepsilon^b} \begin{pmatrix} & & & & 0 \\ & & & \rho_K & \\ & & \iddots & & \\ & \rho_{-K} & & \rho_0 & \\ & & \iddots & & \\ 0 & & & \rho_{-k} & \\ & & & & \end{pmatrix}.$$

Uncertainty Characterization via Scenarios

- *Step 3*: Apply Cholesky decomposition to $\boldsymbol{G}$ (i.e., compute $\boldsymbol{L}$ such that $\boldsymbol{G} = \boldsymbol{L}\boldsymbol{L}^{\boldsymbol{T}}$) so as to obtain the matrix $\boldsymbol{B}$ of the orthogonal transformation.
- *Step 4*: Simulate $N_T \times N_S$ independent standard normal errors, $\boldsymbol{\xi}$.
- *Step 5*: Cross-correlate the $N_T \times N_S$ standard normal errors generated in the previous step by using the orthogonal transformation.
  Specifically, for two stochastic processes, $\boldsymbol{Y}^a$ and $\boldsymbol{Y}^b$, this cross-correlation process is performed as

where n is the sample size of the available historical data set, ε^a_{n-K+1}, $\ldots$, ε^a_n, ε^b_{n-K+1}, $\ldots$, ε^b_n are the last K error terms of the historical series, $\hat{\varepsilon}^a_{n-K+1}$, $\ldots$, $\hat{\varepsilon}^a_n$, $\hat{\varepsilon}^b_{n-K+1}$, $\ldots$, $\hat{\varepsilon}^b_n$ are their corresponding transformed values, and ξ^a_{n+1}, $\ldots$, $\xi^a_{n+N_T}$, ξ^b_{n+1}, $\ldots$, $\xi^b_{n+N_T}$ are the $N_T \times N_S$ ($N_S = 2$) independent standard normal errors simulated in Step 4 above.
Finally, the error series ε^a_{n+1}, $\ldots$, $\varepsilon^a_{n+N_T}$ and ε^b_{n+1}, $\ldots$, $\varepsilon^b_{n+N_T}$ are those to be introduced in the ARMA models fitted in Step 1.

Uncertainty Characterization via Scenarios

- *Step 6*: For each stochastic process, use, as explained in Subsection 3.2.2, the corresponding N_T cross-correlated normal errors to simulate N_T values by means of the corresponding univariate ARMA model fitted in Step 1.
- *Step 7*: Repeat Steps 4-6 until the desired number (N_Ω) of scenarios is obtained.

Uncertainty Characterization via Scenarios

Example (Contemporaneous or quasi-contemporaneous stochastic processes. Algorithm)

Consider two contemporaneous stochastic processes, $\mathbf{Y}^a$ and $\mathbf{Y}^b$, with a contemporaneous correlation coefficient ρ_0 equal to 0.9. Suppose that $y_0^a = 4.10$ and $y_0^b = 2.95$. The procedure to generate a pair of correlated two-period scenarios for these stochastic processes is as follows:

- Step 1 : We fit an ARMA model to each stochastic process by using available historical data. Let us assume that the fitting process results in the following two AR(1) models:

$$y_t^a = 0.910 y_{t-1}^a + \varepsilon_t^a,$$
$$y_t^b = 0.898 y_{t-1}^b + \varepsilon_t^b,$$

$$\begin{pmatrix} y_t^a \\ y_t^b \end{pmatrix} = \begin{pmatrix} 0.910 & 0 \\ 0 & 0.898 \end{pmatrix} \begin{pmatrix} y_{t-1}^a \\ y_{t-1}^b \end{pmatrix} + \begin{pmatrix} \varepsilon_t^a \\ \varepsilon_t^b \end{pmatrix},$$

Uncertainty Characterization via Scenarios

where the estimated standard deviations of residuals, σ_{ε^a} and σ_{ε^b}, are 0.1 and 0.05, respectively.

- Step 2 : Next, we build the variance-covariance matrix **G**,

$$\boldsymbol{G}_{11} = 0.1^2 \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix},$$

$$\boldsymbol{G}_{22} = 0.05^2 \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.0025 & 0 \\ 0 & 0.0025 \end{pmatrix},$$

$$\boldsymbol{G}_{12} = \boldsymbol{G}_{21}^T = 0.1 \times 0.05 \times \begin{pmatrix} 0.9 & 0 \\ 0 & 0.9 \end{pmatrix} = \begin{pmatrix} 0.0045 & 0 \\ 0 & 0.0045 \end{pmatrix}.$$

$$\boldsymbol{G} = \left(\begin{array}{cc|cc} 0.01 & 0 & 0.0045 & 0 \\ 0 & 0.01 & 0 & 0.0045 \\ \hline 0.0045 & 0 & 0.0025 & 0 \\ 0 & 0.0045 & 0 & 0.0025 \end{array}\right).$$

Note that NT = 2 and K = 0.

Uncertainty Characterization via Scenarios

- Step 3 : We apply the Cholesky decomposition to matrix **G** in order to obtain the orthogonal matrix **L**,

$$\boldsymbol{L} = \left(\begin{array}{cc|cc} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ \hline 0.045 & 0 & 0.0218 & 0 \\ 0 & 0.045 & 0 & 0.0218 \end{array}\right).$$

Step 4 : The next step is to simulate $N_T \times N_S = 2 \times 2 = 4$ Independent standard normal errors, $\boldsymbol{\xi}$.

,

$$\boldsymbol{\xi} = \left(\frac{\boldsymbol{\xi}^a}{\boldsymbol{\xi}^b}\right) = \left(\begin{array}{c} \xi_1^a \\ \xi_2^a \\ \hline \xi_1^b \\ \xi_2^b \end{array}\right) = \left(\begin{array}{c} -0.4326 \\ -1.6656 \\ \hline 0.1253 \\ 0.2877 \end{array}\right).$$

Uncertainty Characterization via Scenarios

- Step 5 : We cross-correlate the four standard normal errors generated in the previous step by using the orthogonal transformation.

$$\begin{pmatrix} \varepsilon_1^a \\ \varepsilon_2^a \\ \hline \varepsilon_1^b \\ \varepsilon_2^b \end{pmatrix} = \left(\begin{array}{cc|cc} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ \hline 0.045 & 0 & 0.0218 & 0 \\ 0 & 0.045 & 0 & 0.0218 \end{array}\right) \begin{pmatrix} -0.4326 \\ -1.6656 \\ \hline 0.1253 \\ 0.2877 \end{pmatrix} = \begin{pmatrix} -0.0433 \\ -0.1666 \\ \hline -0.0167 \\ -0.0687 \end{pmatrix}.$$

Step 6 : Finally, we use the cross-correlated normal errors generated in Step 5 above to simulate two values of stochastic processes $\boldsymbol{Y}^a$ and $\boldsymbol{Y}^b$ by means of the ARMA models fitted in Step 1, that is,

$$\begin{aligned} \begin{pmatrix} y_1^a \\ y_1^b \end{pmatrix} &= \begin{pmatrix} 0.910 & 0 \\ 0 & 0.898 \end{pmatrix} \begin{pmatrix} y_0^a \\ y_0^b \end{pmatrix} + \begin{pmatrix} \varepsilon_1^a \\ \varepsilon_1^b \end{pmatrix} \\ &= \begin{pmatrix} 0.910 & 0 \\ 0 & 0.898 \end{pmatrix} \begin{pmatrix} 4.10 \\ 2.95 \end{pmatrix} + \begin{pmatrix} -0.0433 \\ -0.0167 \end{pmatrix} = \begin{pmatrix} 3.69 \\ 2.63 \end{pmatrix}; \end{aligned}$$

Uncertainty Characterization via Scenarios

$$\begin{pmatrix} y_2^a \\ y_2^b \end{pmatrix} = \begin{pmatrix} 0.910 & 0 \\ 0 & 0.898 \end{pmatrix} \begin{pmatrix} y_1^a \\ y_1^b \end{pmatrix} + \begin{pmatrix} \varepsilon_2^a \\ \varepsilon_2^b \end{pmatrix}$$

$$= \begin{pmatrix} 0.910 & 0 \\ 0 & 0.898 \end{pmatrix} \begin{pmatrix} 3.69 \\ 2.63 \end{pmatrix} + \begin{pmatrix} -0.1666 \\ -0.0687 \end{pmatrix} = \begin{pmatrix} 3.19 \\ 2.29 \end{pmatrix}.$$

In summary, the pair of correlated scenarios $\left(\hat{\boldsymbol{Y}}^a, \hat{\boldsymbol{Y}}^b\right)^T$ covering two periods are

$$\begin{pmatrix} \hat{\boldsymbol{Y}}^a \\ \hat{\boldsymbol{Y}}^b \end{pmatrix} = \begin{pmatrix} 3.69 & 3.19 \\ 2.63 & 2.29 \end{pmatrix}.$$

Uncertainty Characterization via Scenarios

❑ Scenarios for non-contemporaneous stochastic processes

- This is the case, for instance, of the pool price and the system demand. In fact, pool price and demand are correlated contemporaneously, but they also present a non-negligible cross-correlation in lags significantly larger than 1.
- In this subsection we propose a methodology based on dynamic regression models that can be efficiently used to generate scenarios for non-contemporaneous stochastic processes.
- We define $\boldsymbol{Y}$ $\{y_t,\ t \in T\}$ and $\boldsymbol{Y}^e$ $\{y_t^e,\ t \in T\}$ as two non-contemporaneous processes, where $\boldsymbol{Y}^e$ is considered an explanatory process of $\boldsymbol{Y}$. This means that the behavior of Y can be explained in part by the knowledge of $\boldsymbol{Y}^e$.

$$y_t = c + \sum_{j=1}^{p_1} u_j y_{t-j} + \sum_{j=0}^{p_2} v_j y_{t-j}^e + \varepsilon_t,$$

$$y_t^e = c^e + \sum_{j=1}^{p_e} \phi_j^e y_{t-j}^e + \varepsilon_t^e - \sum_{q=1}^{q_e} \theta_j^e \varepsilon_{t-j}^e,$$

Uncertainty Characterization via Scenarios

✓ Given that the ARMA model for the explanatory process and the dynamic regression model for the forecast process are already established, the scenario generation process is outlined below.

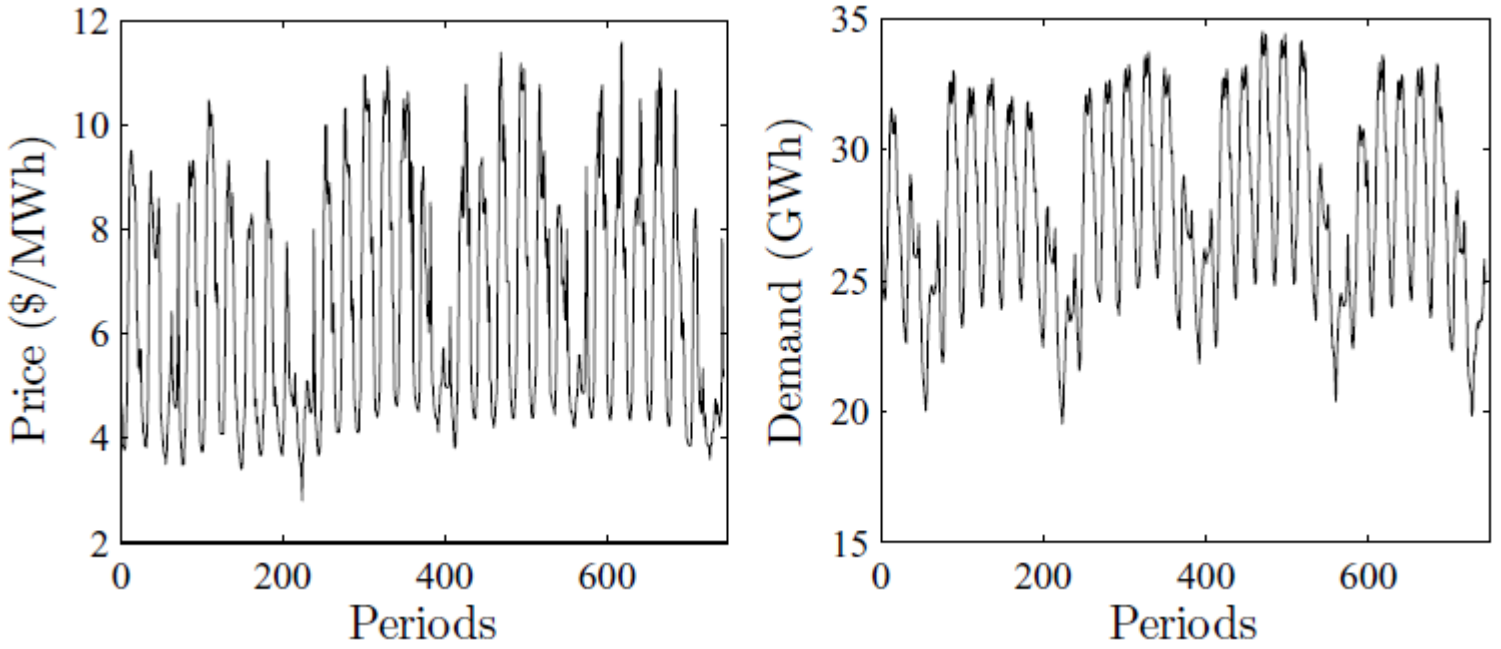
❑ Algorithm

- *Step 1*: Initialize the scenario counter: $\omega \leftarrow 0$.
- *Step 2*: Update the scenario counter and initialize the time period counter: $\omega \leftarrow \omega + 1$, $t \leftarrow 0$.
- *Step 3*: Update the time period counter: $t \leftarrow t + 1$.
- *Step 4*: Randomly generate $\varepsilon_t^e(\omega) \sim \mathrm{N}(0, \sigma_\varepsilon^e)$.
- *Step 5*: Evaluate $y_t^e(\omega)$.
- *Step 6*: Randomly generate $\varepsilon_t(\omega) \sim \mathrm{N}(0, \sigma_\varepsilon)$.
- *Step 7*: Evaluate $y_t(\omega)$.
- *Step 8*: If $t < N_T$ go to Step 3), else go to Step 9).
- *Step 9*: If $\omega < N_\Omega$ go to Step 2), else the scenario-generation process concludes.

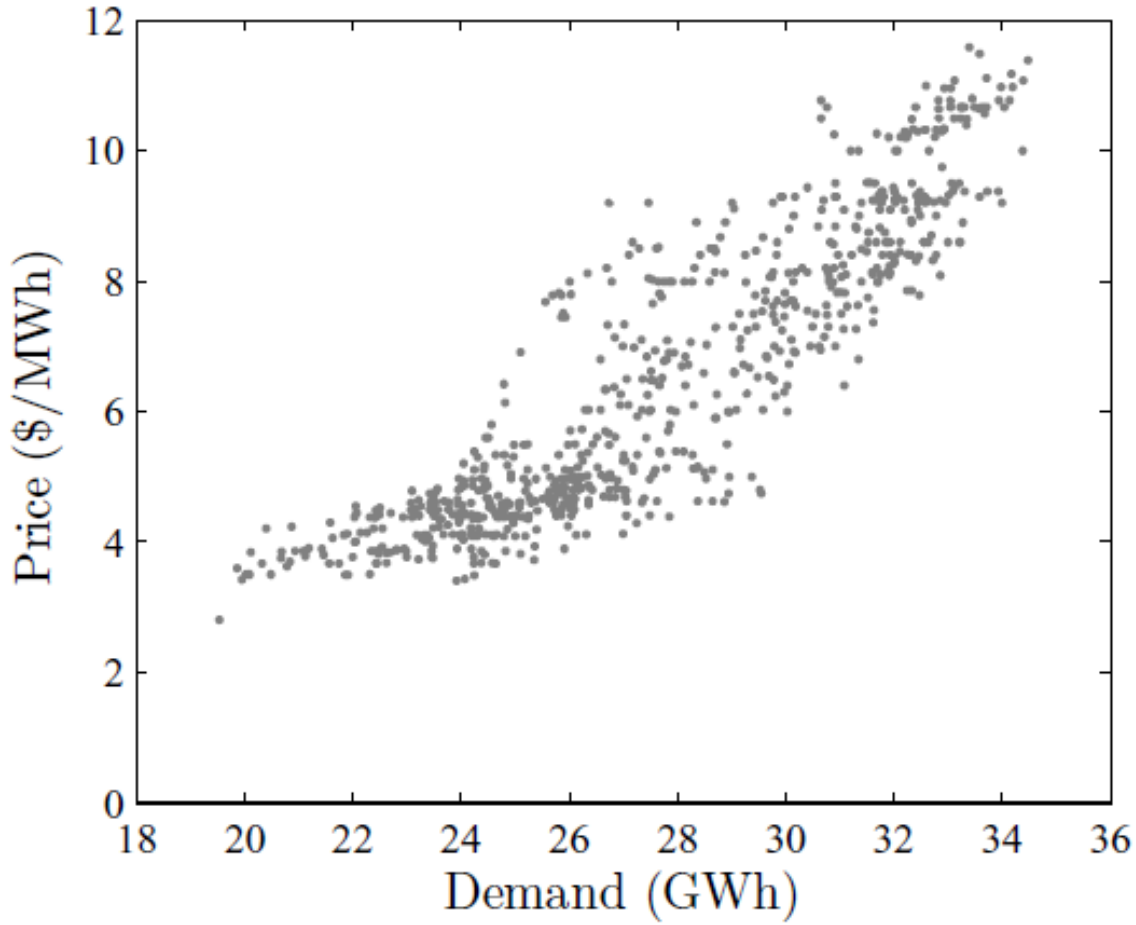
Uncertainty Characterization via Scenarios

❑ Scenario Generation Using ARIMA and Dynamic Regression models: Electricity Prices and Demand

- Observe that daily and weekly seasonalities can be identified in the historical data of both series. Moreover, at first sight, pool prices and demand are strongly correlated.
- Next, pool price and demand scenarios for one week are generated independently using ARIMA models without modeling explicitly the correlation between both series.



Uncertainty Characterization via Scenarios



Scenario-generation case studies: correlation between pool prices and demand

Uncertainty Characterization via Scenarios

$$\left(1-\phi_1^{\mathrm{P}}B-\phi_2^{\mathrm{P}}B^2\right)\left(1-\phi_{24}^{\mathrm{P}}B^{24}\right)\left(1-\phi_{168}^{\mathrm{P}}B^{168}\right)\log\left(\lambda_t^{\mathrm{P}}\right)=$$
$$\left(1-\theta_1^{\mathrm{P}}B-\theta_2^{\mathrm{P}}B^2\right)\left(1-\theta_{168}^{\mathrm{P}}B^{168}\right)\varepsilon_t^{\mathrm{P}},$$

where λ_t^{P} represents the pool price in period t, and $\varepsilon_t^{\mathrm{P}} \sim N(0,\sigma^{\mathrm{P}})$ is the error term. Observe that the logarithm function is used to stabilize the variance of the series.
In a similar manner, the ARIMA model describing the demand is

$$\left(1-\phi_1^{\mathrm{D}}B\right)\left(1-\phi_{24}^{\mathrm{D}}B^{24}\right)\left(1-\phi_{168}^{\mathrm{D}}B^{168}\right)\log\left(E_t^{\mathrm{D}}\right)=$$
$$\left(1-\theta_{168}^{\mathrm{D}}B^{168}\right)\varepsilon_t^{\mathrm{D}},$$

where E_t^{D} represents the demand in period t, and $\varepsilon_t^{\mathrm{D}} \sim N(0,\sigma^{\mathrm{D}})$ is the error term.

Uncertainty Characterization via Scenarios

The adjusted parameters of models are provided in Table.

Table 3.3 Scenario-generation case studies: parameters for the price and demand ARIMA models

Price		Demand	
ϕ_1^{P}=0.3135	θ_1^{P}=-0.4495	ϕ_1^{D}=0.9470	$\theta_{168}^{\mathrm{D}}$= 0.6027
ϕ_2^{P}=0.6580	θ_2^{P}=0.1356	ϕ_{24}^{D}=0.3449	
ϕ_{24}^{P}=0.2599	$\theta_{168}^{\mathrm{P}}$=0.4717	ϕ_{168}^{D}=0.9813	
ϕ_{168}^{P} = 0.8674			

Using the scenario-generation algorithm described, for both pool price and demand series, a set of 1000 scenarios for each series is obtained.

Uncertainty Characterization via Scenarios

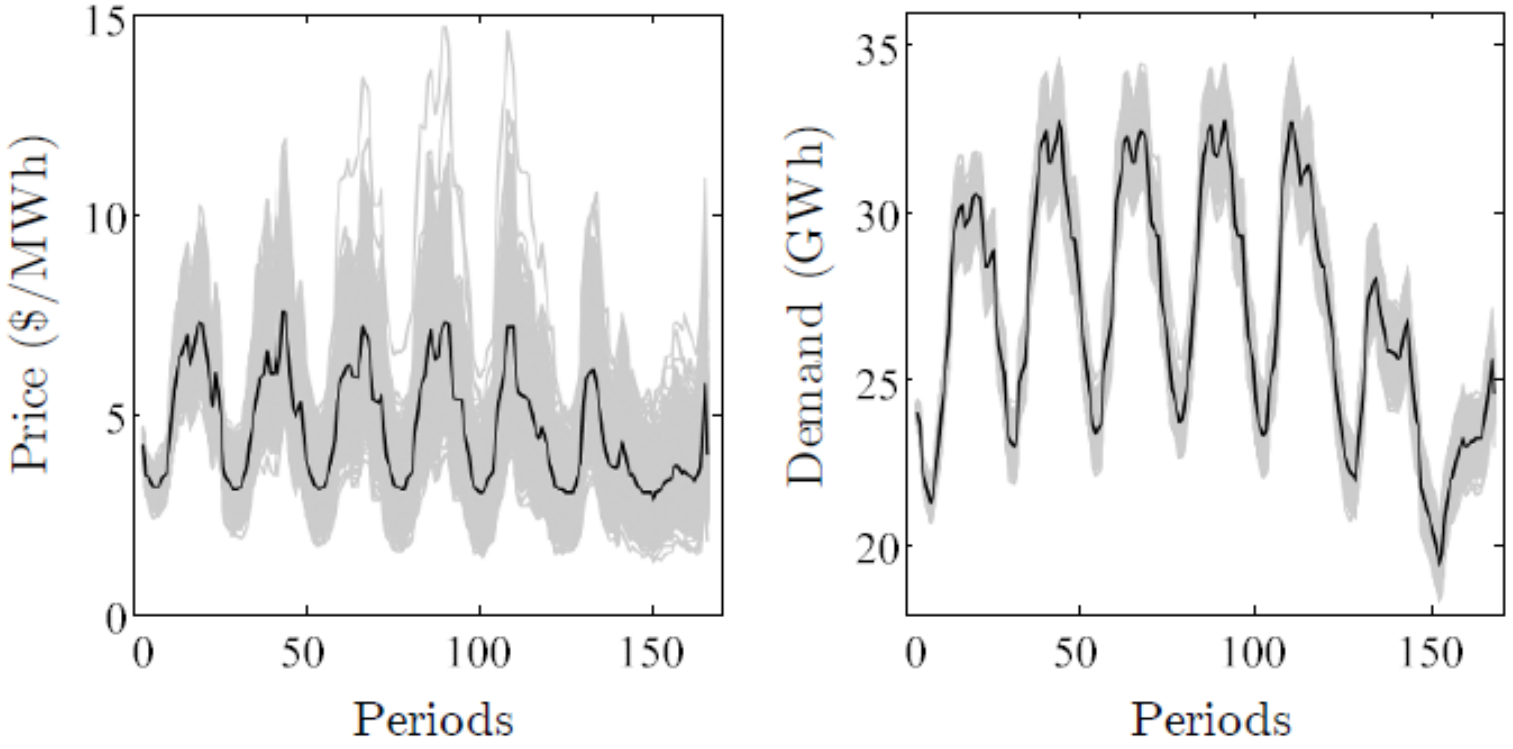


Fig. 3.13 Scenario-generation case studies: price and demand scenarios using ARIMA models

Uncertainty Characterization via Scenarios

Once the scenarios of price and demand are obtained, it is possible to plot conveniently their values to observe the correlation between both series.

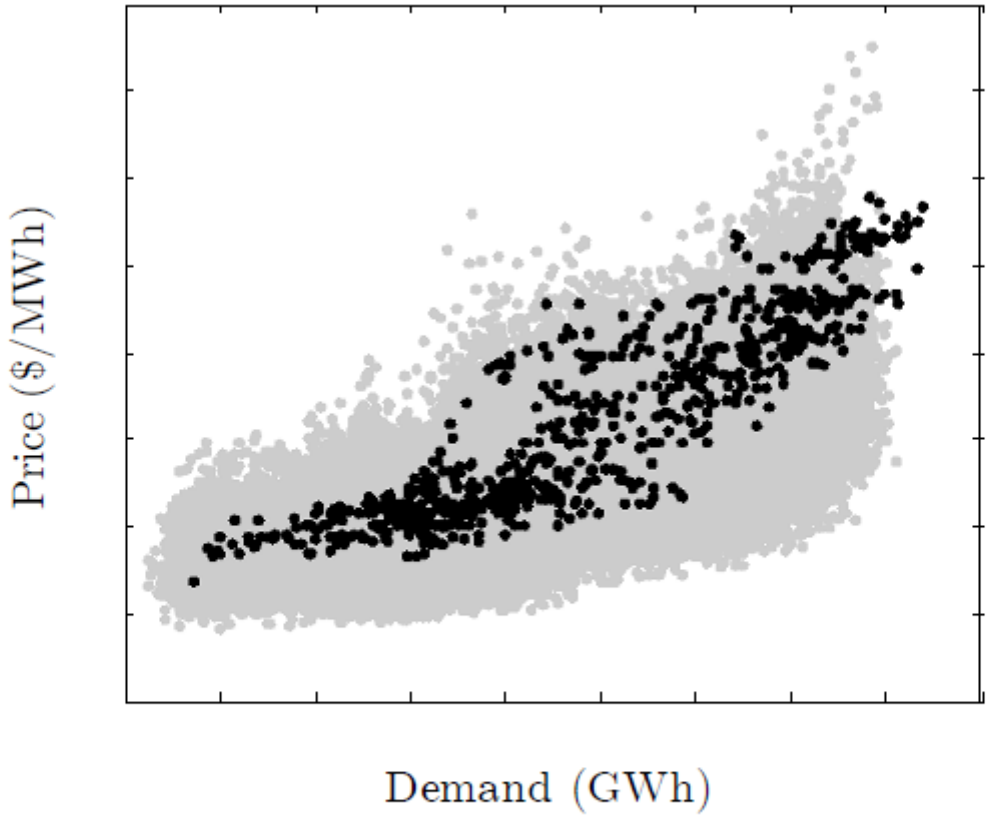


Fig. 3.14 Scenario-generation case studies: correlation of pool price and demand scenarios using ARIMA models

- Black points represent the pairs price-demand observed in the historical data, whereas grey points correspond to the pairs resulting from the generated scenarios.

Uncertainty Characterization via Scenarios

- The dynamic regression model proposed is

$$\left(1 - u_1^{\mathrm{R}} B - u_2^{\mathrm{R}} B^2\right)\left(1 - u_{24}^{\mathrm{R}} B^{24}\right)\left(1 - u_{168}^{\mathrm{R}} B^{168}\right)\log\left(\lambda_t^{\mathrm{P}}\right) = \left(v_0^{\mathrm{R}} + v_1^{\mathrm{R}} B + v_2^{\mathrm{R}} B^2\right)\log\left(E_t^{\mathrm{D}}\right) + \varepsilon_t^{\mathrm{R}},$$

The parameters of the dynamic regression model:

Table 3.4 Scenario-generation case studies: price dynamic regression model parameters

u_1^{R}=0.7073	v_0^{R}= 1.5514	
u_2^{R}=0.1306	v_1^{R}=-1.0658	σ^{R} is 0.0295
u_{24}^{R}= 0.1998	v_2^{R}=-0.3172	
u_{168}^{R} = 0.2901		

Uncertainty Characterization via Scenarios

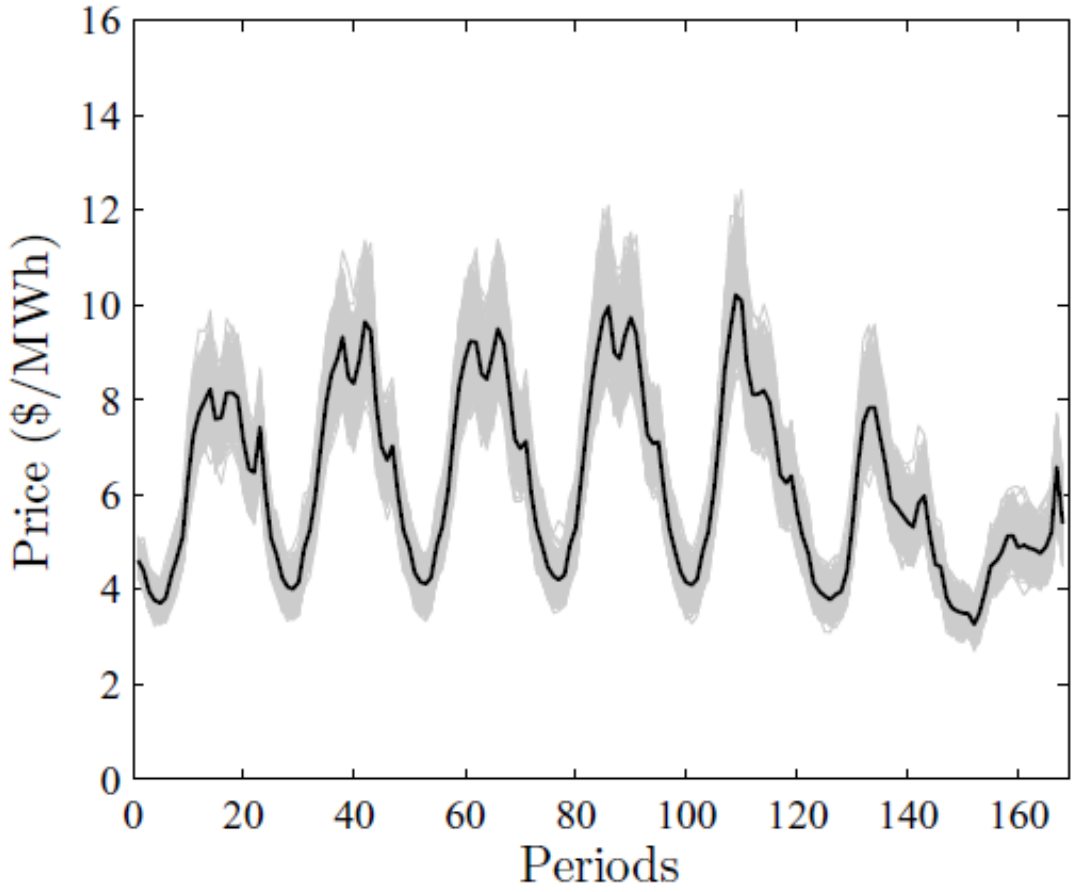


Fig. 3.16 Scenario-generation case studies: pool price scenarios using a dynamic regression model