Stochastic Programming Fundamentals

Stochastic Programming Fundamentals

- Unknown data abound in decision-making problems in the real world. This lack of perfect information is common in problems belonging to different knowledge areas such as engineering, economics, finances, etc.

- For example, electricity prices are unknown when agents have to submit their offers or bids to the pool. Similarly, at the time of procuring the energy needed to supply client loads, retailers do not know precisely the electricity demands of these clients.

- This is what motivates the use of *stochastic programming* models for decision making under uncertainty.

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- ✓ **If the input data of an optimization problem are well-defined and deterministic, its optimal solution (decision) is achieved by solving the problem.**

- ✓ **If not, it is not clear how the decision-making problem should be formulated. One possibility is to substitute the uncertain input data (describable through probability functions) by their corresponding expected values, which results in a well defined and deterministic optimization problem.**

Alternatively, the probability distribution of input data can be approximated by a collection of plausible sets of input data with associated probabilities of occurrence.

Then a stochastic optimization problem can be formulated implicitly weighting (with the probabilities of occurrence) the individual solutions associated with each set of input data to achieve a single solution that is the best in some sense for all sets of input data. Hence, we achieve a solution that is adequately pre-positioned with respect to all the sets of input data, but not to any one of them particularly.

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- As a result of the uncertain input data being described by a collection of different sets of data, the resulting objective function is uncertain and needs to be characterized as a random variable. Since such objective function is not a real-valued function but a random variable, the problem of establishing a specific objective for the decision-making problem arises.

- One alternative is to maximize the expected value of the objective function, other one, to maximize the expected value of such function but limiting its variance, etc.

- The solution of stochastic programming is not the best for each individual set of input data but it is the best if all of them, weighted with their probabilities of occurrence, are simultaneously considered.

- The price to be paid for using a stochastic programming approach is a dramatic increase in the size of the problem to be solved, which if handled without care may lead to intractability.

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❑ Random Variables

Within a stochastic programming context, each uncertain parameter is modeled as a random variable.

For instance, random variable λ can be represented by $\lambda(\omega),\ \omega = 1, \ldots, N_\Omega$,where ω is the scenario index, N_Ω is the number of scenarios considered and Ω is the set of scenarios.

We denote by λ_Ω the set of possible realizations of random variable λ , i.e., $\lambda_\Omega = \{\lambda(1), \ldots, \lambda(N_\Omega)\}$

- ✓ Each realization $\lambda(\omega)$ is associated with a probability $\pi(\omega)$ defined as

$$\pi(\omega) = P(\omega|\lambda = \lambda(\omega)), \quad \text{where} \quad \sum_{\omega \in \Omega} \pi(\omega) = 1$$

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Example (Random variable: pool price at noon)

The pool price at noon on a future day of a particular electricity market (e.g., ISO New England) is a random variable.

$$\$50/\text{MWh with probability } 0.2$$
$$\$46/\text{MWh with probability } 0.6$$
$$\$44/\text{MWh with probability } 0.2\ .$$

Assuming a finite set of scenarios, random variable λ can be characterized by its cumulative distribution function (cdf).

$$F_\lambda(\eta) = P(\omega|\lambda(\omega) \leq \eta) = \sum_{\omega \in \Omega | \lambda(\omega) \leq \eta} \pi(\omega),\ \forall \eta \in \mathbb{R},$$

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A random variable can also be characterized by its statistical moments. Two useful moments are the mean (expected value) and the variance. The mean and the variance of random variable λ , $\overline{\lambda}$ and σ_{λ}^{2} ,respectively, are

$$\overline{\lambda} = \mathcal{E}\{\lambda\} = \sum_{\omega \in \Omega} \pi(\omega)\lambda(\omega)$$

$$\sigma_{\lambda}^{2} = \mathcal{V}\{\lambda\} = \sum_{\omega \in \Omega} \pi(\omega)\Big(\lambda(\omega) - \mathcal{E}\{\lambda\}\Big)^{2}$$

The mean is the expected value of the random variable, whereas the variance is a dispersion measure, whose square root is the standard deviation. Hereinafter we use the standard deviation instead of the variance to characterize the dispersion of a random variable.

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Example (Pool price at noon: distribution and moments)

$$\overline{\lambda} = 0.2 \times 50 + 0.6 \times 46 + 0.2 \times 44 = \$46.40/\text{MWh}$$

$$\sigma_\lambda^2 = 0.2 \times (50 - 46.4)^2 + 0.6 \times (46 - 46.4)^2 + 0.2 \times (44 - 46.4)^2$$

$$= 3.84\ (\$/\text{MWh})^2$$

$$\sigma_\lambda = \sqrt{0.2 \times (50 - 46.4)^2 + 0.6 \times (46 - 46.4)^2 + 0.2 \times (44 - 46.4)^2}$$

$$= \$1.9596/\text{MWh}\ .$$

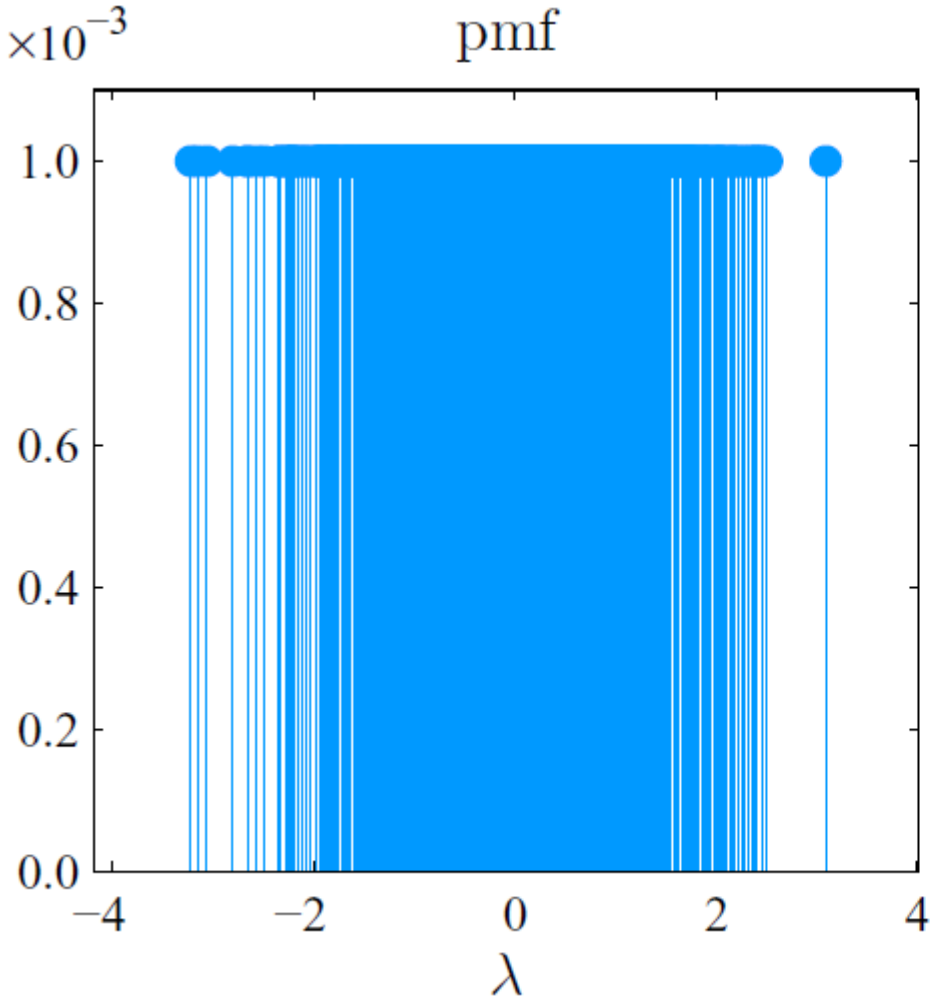
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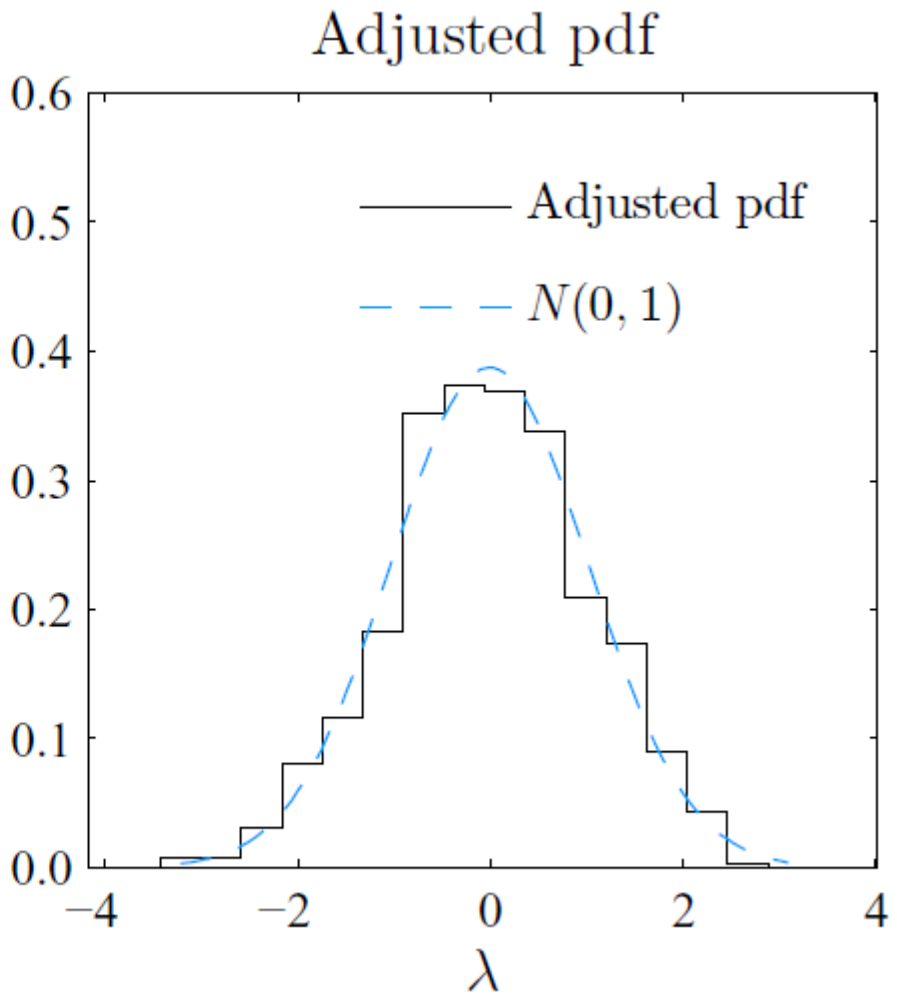
- A discrete random variable is also statistically described by its probability mass function (pmf). The pmf gives the probability of a discrete random variable being equal to a given value. The equivalent of the pmf for continuous variables is the probability density function (pdf).

From a graphical point of view, the pmf is not appropriate for representing discrete random variables with many realizations, as the resulting graphic may be difficult to interpret. Therefore, we propose the use of an alternative representation referred to as adjusted pdf. The adjusted pdf of a discrete random variable λ is a bar chart built as follows:

1. The base of each bar represents a range of values of the random variable.
2. The height of each bar is equal to the sum of the probabilities of all realizations falling within the interval spanned by the base of the bar divided by the width of the bar.

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❑ Stochastic Processes

A random variable whose value evolves over time is known as a stochastic process.

- A stochastic process is constituted by a set of dependent random variables sequentially arranged in time.
- For each time period, the corresponding random variable (e.g., the price at noon) depends on the other random variables (e.g., the prices in other hours of the day).
- Stochastic processes spanning a given time horizon can be represented by scenarios.

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For instance, stochastic process $\boldsymbol{\lambda}$ can be represented by vectors $\boldsymbol{\lambda}(\omega)$, $\omega = 1, \ldots, N_\Omega$, where ω is the scenario index and N_Ω is the number of scenarios considered. We denote by $\boldsymbol{\lambda}_\Omega$ the set of possible realizations of stochastic process $\boldsymbol{\lambda}$, i.e., $\boldsymbol{\lambda}_\Omega = \{\boldsymbol{\lambda}(1), \ldots, \boldsymbol{\lambda}(N_\Omega)\}$

- ✓ Each realization $\boldsymbol{\lambda}(\omega)$ is associated with a probability $\pi(\omega)$ defined as

$$\pi(\omega) = P(\omega|\boldsymbol{\lambda} = \boldsymbol{\lambda}(\omega)), \quad \text{where} \quad \sum_{\omega \in \Omega} \pi(\omega) = 1.$$

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Example (Stochastic process: hourly pool prices at two consecutive hours)

We consider that the random variable representing the pool price at noon is characterized as

$$\$50/\text{MWh with probability } 0.2$$
$$\$46/\text{MWh with probability } 0.6$$
$$\$44/\text{MWh with probability } 0.2$$

The random variable representing the pool price at 1pm is described in the following:

1. If the pool price at noon is $50/MWh, the probabilities of the price at 1 pm being 52, 48 or $45/MWh are, respectively, 0.8, 0.1 or 0.1.
2. If, on the other hand, the pool price at noon is $46/MWh, the probabilities of the price at 1 pm being 52, 48 or $45/MWh are, respectively, 0.2, 0.7 or 0.1.
3. Finally, if the pool price at noon is $44/MWh, the probabilities of the price at 1 pm being 52, 48 or $45/MWh are, respectively, 0.1, 0.4 or 0.5.

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❑ Scenarios

- From a computational viewpoint, a convenient manner to characterize stochastic processes is through scenarios.

- To adequately describe a stochastic process, it is critical to generate a vey large number of scenarios so that these scenarios cover the most plausible realizations of the considered stochastic process, which makes stochastic programming problem computationally intractable.
- So procedures are required to reduce the numbers of scenario in a way retain most of the relevant information on the stochastic process contained in the original scenario set while reducing significantly its cardinality.

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Example (Stochastic process: scenarios)

Illustrative Example 2.5: price scenarios

Scenario #	Price at noon ($/MWh)	Price at 1pm ($/MWh)	Probability (per unit)
1	50	52	$0.2 \times 0.8 = 0.16$
2	50	48	$0.2 \times 0.1 = 0.02$
3	50	45	$0.2 \times 0.1 = 0.02$
4	46	52	$0.6 \times 0.2 = 0.12$
5	46	48	$0.6 \times 0.7 = 0.42$
6	46	45	$0.6 \times 0.1 = 0.06$
7	44	52	$0.2 \times 0.1 = 0.02$
8	44	48	$0.2 \times 0.4 = 0.08$
9	44	45	$0.2 \times 0.5 = 0.10$

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Stochastic Programming Problems

- In decision making under uncertainty, the decision maker has to make optimal decisions throughout a decision horizon with incomplete information.
- Over the considered decision horizon, a number of stages is defined.
- Each stage represents a point in time where decisions are made or where uncertainty partially or totally vanishes.
- The amount of information available to the decision maker is usually different from stage to stage.

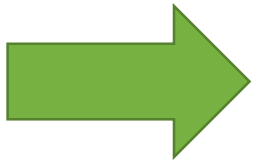
Stochastic Programming Fundamentals

❑ Two-Stage Problems

- We consider a decision-making problem where decisions are made at two stages and there exists a stochastic process $\boldsymbol{\lambda}$ represented by a set of scenarios $\boldsymbol{\lambda}_{\Omega}$. We assume that two different decision variable vectors, namely x and y, are involved in this problem.

- Decision x is made before knowing the actual value of the stochastic process $\boldsymbol{\lambda}$, while y is determined after the realization of $\boldsymbol{\lambda}$.

- Consequently, decision y depends on the decision x previously made and on the realization $\boldsymbol{\lambda}(\omega)$ of the stochastic process $\boldsymbol{\lambda}$.



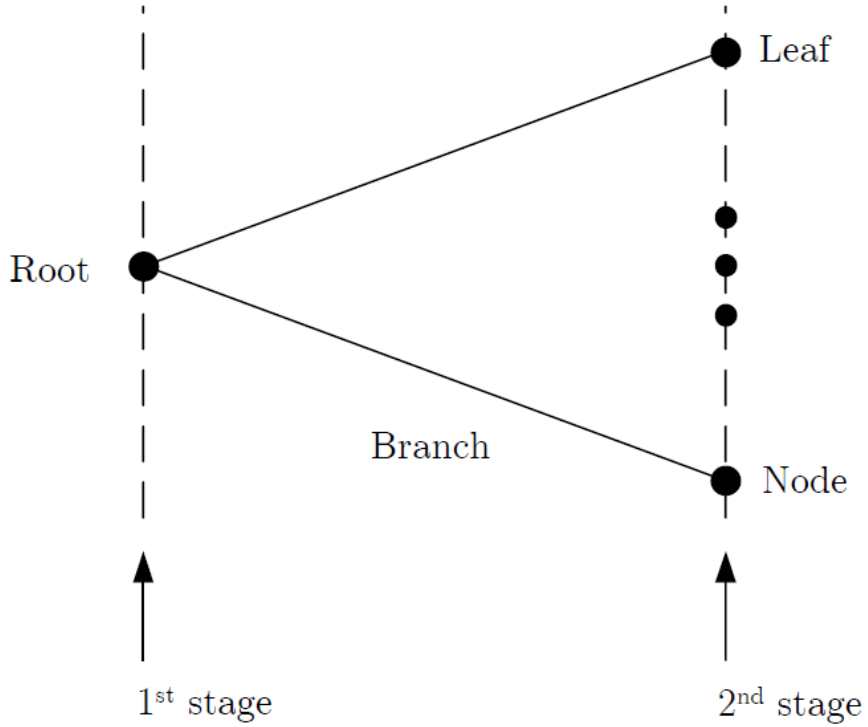
1. Decision $\boldsymbol{x}$ is made.
2. The stochastic process $\boldsymbol{\lambda}$ is realized as $\boldsymbol{\lambda}(\omega)$.
3. Decision $\boldsymbol{y}(\boldsymbol{x}, \omega)$ is made.

Stochastic Programming Fundamentals

- **In this decision-making process, two different kinds of decisions are distinguished:**

1. First-stage or here-and-now decisions. These decisions are made before the realization of the stochastic process. Hence, variables representing here-and-now decisions do not depend on each realization of the stochastic process.

2. Second-stage or wait-and-see decisions. These decisions are made after knowing the actual realization of the stochastic process. Consequently, these decisions depend on each realization vector of the stochastic process. If the stochastic process is represented by a set of scenarios, a second stage decision variable is defined for each single scenario considered.



Stochastic Programming Fundamentals

- ✓ It is important to note that stochastic programming problems can be mathematically formulated using either a node-variable formulation or a scenario-variable formulation.

- ✓ The general expression of a two-stage stochastic linear programming problem is the following:

$$
\begin{aligned}
&\text{Minimize}_{\boldsymbol{x}} \;\; z = \; \boldsymbol{c}^{\top}\boldsymbol{x} + \mathcal{E}\{Q(\omega)\} \\
&\text{subject to } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
&\qquad\qquad \boldsymbol{x} \in X \\
&\text{where} \\
&Q(\omega) = \Big\{ \text{Minimize}_{\boldsymbol{y}(\omega)} \;\; \boldsymbol{q}(\omega)^{\top}\boldsymbol{y}(\omega) \\
&\qquad\qquad \text{subject to } \boldsymbol{T}(\omega)\boldsymbol{x} + \boldsymbol{W}(\omega)\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega) \\
&\qquad\qquad\qquad\qquad \boldsymbol{y}(\omega) \in Y \Big\}, \;\; \forall \omega \in \Omega,
\end{aligned}
$$

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Or

$$\text{Minimize}_{\boldsymbol{x},\boldsymbol{y}(\omega)} \quad z = \boldsymbol{c}^{\top}\boldsymbol{x} + \sum_{\omega\in\Omega} \pi(\omega)\boldsymbol{q}(\omega)^{\top}\boldsymbol{y}(\omega)$$

$$\begin{aligned} \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{T}(\omega)\boldsymbol{x} + \boldsymbol{W}(\omega)\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega),\ \forall\omega \in \Omega \\ & \boldsymbol{x} \in X,\ \boldsymbol{y}(\omega) \in Y,\ \forall\omega \in \Omega. \end{aligned}$$

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- The first formulation relies on variables associated with decision points while the second one relies on variables associated with scenarios.
- The first formulation is comparatively more compact than the second one and is particularly well suited for a direct solution approach.
- The second one requires a larger number of variables and constraints than the first one but presents an exploitable structure that is well suited for decomposition.

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Example (Two-stage problem. Node-variable formulation)

An electricity consumer is facing both uncertain electricity demand and price for next week. For simplicity, we consider that both price and demand are uncertain but constant throughout the week.

Illustrative Example 2.6: scenario data for the consumer

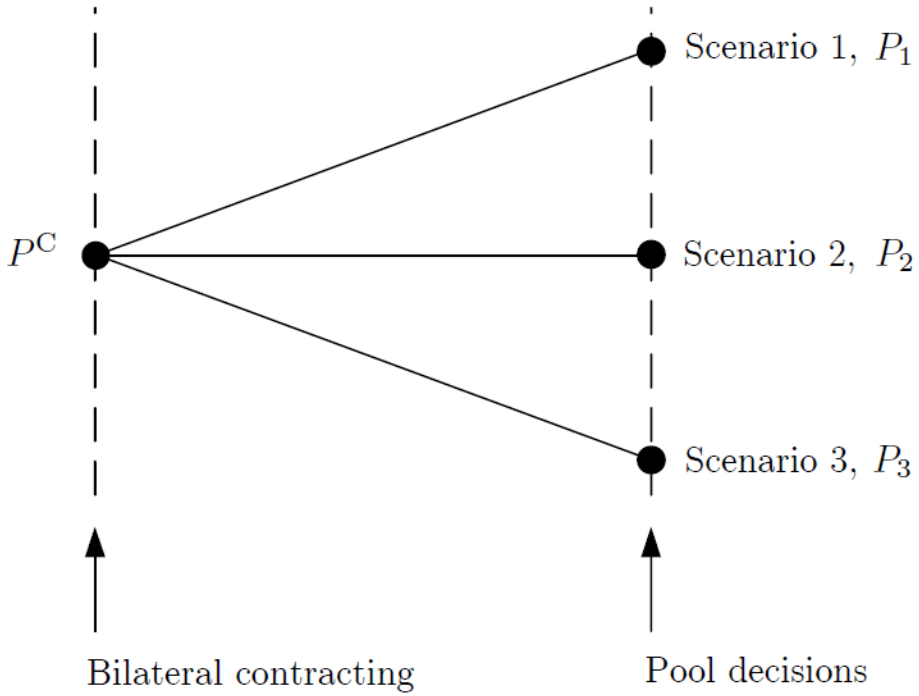
Scenario #	Probability (per unit)	Demand (MW)	Price ($/MWh)
1	0.2	110	50
2	0.6	100	46
3	0.2	80	44

Additionally, this consumer has the possibility of buying up to 90 MW at $45/MWh throughout next week, by signing a bilateral contract before next week, i.e., before knowing the actual demand and pool price it has to face.

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The decision-making problem of this consumer can be formulated as a two-stage stochastic programming problem:

At the first stage, the consumer has to decide how much to buy from the contract. The second stage reproduces pool purchases for each of the three considered demand/price realizations (scenarios).



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✓ The node-variable formulation of this two-stage stochastic programming problem is as follows:

$$\text{Minimize}_{P^{\mathrm{C}},P_1,P_2,P_3}$$

$$C^{\mathrm{S}} = 168(45P^{\mathrm{C}} + 0.2 \times 50P_1 + 0.6 \times 46P_2 + 0.2 \times 44P_3)$$

$$\text{subject to}$$

$$P^{\mathrm{C}} + P_1 \geq 110$$

$$P^{\mathrm{C}} + P_2 \geq 100$$

$$P^{\mathrm{C}} + P_3 \geq 80$$

$$0 \leq P^{\mathrm{C}} \leq 90$$

$$0 \leq P_1, P_2, P_3 \ .$$

$$P^{\mathrm{C}*} = 80$$
$$P_1^* = 30$$
$$P_2^* = 20$$
$$P_3^* = 0$$

- Variable Pc represents the power bought through the bilateral contract, while variables P1, P2, and P3 represent the power bought in the pool for scenarios 1, 2, and 3, respectively.

Example (Two-stage problem. Scenario-variable formulation)

The problem in previous example expressed according to a scenario variable representation is formulated as follows:

$$\text{Minimize}_{P_1^{\mathrm{C}},P_2^{\mathrm{C}},P_3^{\mathrm{C}},P_1,P_2,P_3}$$

$$0.2\times 168(45P_1^{\mathrm{C}}+50P_1)+0.6\times 168(45P_2^{\mathrm{C}}+46P_2)+0.2\times 168(45P_3^{\mathrm{C}}+44P_3)$$

$$\text{subject to}$$

$$P_1^{\mathrm{C}}+P_1\geq 110$$

$$P_2^{\mathrm{C}}+P_2\geq 100$$

$$P_3^{\mathrm{C}}+P_3\geq 80$$

$$0\leq P_1^{\mathrm{C}},P_2^{\mathrm{C}},P_3^{\mathrm{C}}\leq 90$$

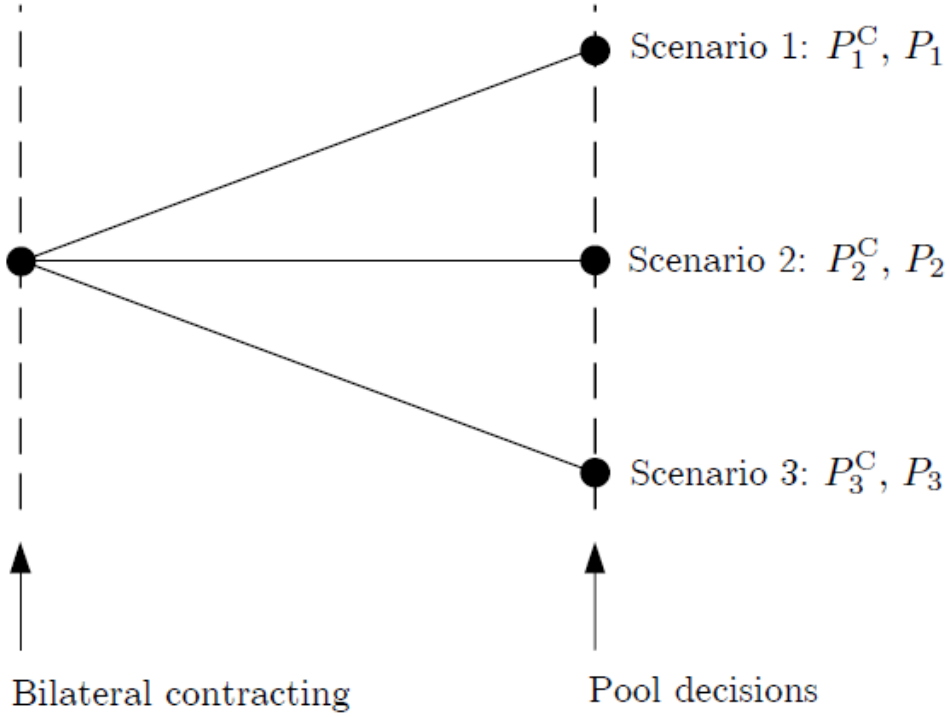
$$0\leq P_1,P_2,P_3$$

$$P_1^{\mathrm{C}}=P_2^{\mathrm{C}}=P_3^{\mathrm{C}}\ .$$

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- Variables Pc1 , Pc2 , and Pc3 represent the power bought through the bilateral contract under scenarios 1, 2, and 3, respectively, while variables P1, P2,and P3 represent the power bought in the pool also in scenarios 1, 2, and 3, respectively.

- The variables of this problem are associated with the scenarios and thus the denomination scenario-variable formulation.
- Generally, a scenario-variable formulation entails a larger number of variables and constraints than a node-variable formulation, but the scenario-variable formulation allows using efficiently solution procedures based on decomposition.



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❑ Multi-Stage Problems

In some cases, decision-making problems comprise more than two stages, and the two-stage stochastic programming.

An example of a multi-stage decision-making process with r stages is the following:

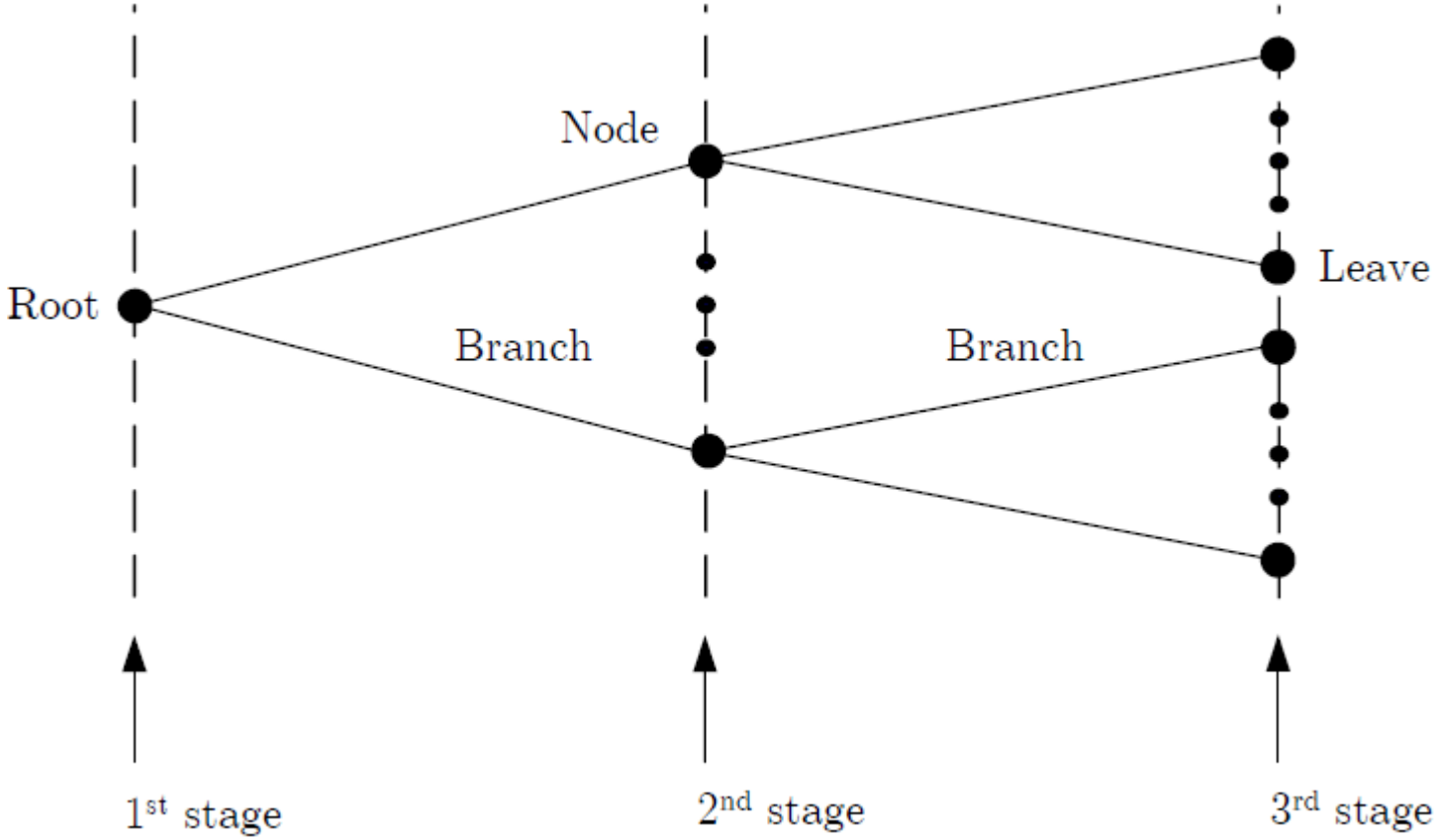
1. Decisions x^1 are made.
2. Stochastic process λ^1 is realized as $\lambda^1(\omega^1)$.
3. Decisions $x^2(x^1, \omega^1)$ are made.
4. Stochastic process λ^2 is realized as $\lambda^2(\omega^2)$.
5. Decisions $x^3(x^1, \omega^1, x^2, \omega^2)$ are made.

. . .

2r-2. Stochastic process λ^{r-1} is realized as $\lambda^{r-1}(\omega^{r-1})$.
2r-1. Decisions $x^r(x^1, \omega^1, \ldots, x^{r-1}, \omega^{r-1})$ are made.

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This decision framework is conveniently visualized through a scenario tree.



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- ✓ In all stochastic programming problems, and particularly in multi-stage problems, it is important to establish the non-anticipativity of the decisions.
- ✓ The non-anticipativity of the decisions is implicitly taken into account in the previous decision sequence.

$$\begin{aligned} \text{Minimize}_{\boldsymbol{x}^1} \quad & z = \boldsymbol{c}^{1,\top}\boldsymbol{x}^1 + \mathcal{E}_{\omega^1}\{Q^1(\boldsymbol{x}^1,\omega^1)\} \\ \text{subject to} \quad & \boldsymbol{A}^1\boldsymbol{x}^1 = \boldsymbol{b}^1 \\ & \boldsymbol{x}^1 \in X^1, \end{aligned}$$

where

$$Q^1(\boldsymbol{x}^1,\omega^1) = \left\{ \begin{aligned} \text{Minimize}_{\boldsymbol{x}^2(\omega^1)} \quad & \boldsymbol{c}^{2,\top}(\omega^1)\boldsymbol{x}^2(\omega^1) + \mathcal{E}_{\omega^2}\{Q^2(\boldsymbol{x}^1,\omega^1,\boldsymbol{x}^2(\omega^1),\omega^2)\} \\ \text{subject to} \quad & \boldsymbol{T}^{1,1}(\omega^1)\boldsymbol{x}^1 + \boldsymbol{T}^{1,2}(\omega^1)\boldsymbol{x}^2(\omega^1) = \boldsymbol{h}^1(\omega^1) \\ & \boldsymbol{x}^2(\omega^1) \in X^2 \end{aligned} \right\}, \ \forall \omega^1 \in \Omega^1,$$

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where

$$
Q^2(\boldsymbol{x}^1, \omega^1, \boldsymbol{x}^2(\omega^1), \omega^2) = \Bigg\{
$$

$$
\begin{aligned}
\text{Minimize}_{\boldsymbol{x}^3(\omega^1,\omega^2)} \quad & \boldsymbol{c}^{3,\top}(\omega^1,\omega^2)\boldsymbol{x}^3(\omega^1,\omega^2) + \\
& \mathcal{E}_{\omega^3}\{Q^3(\boldsymbol{x}^1,\omega^1,\boldsymbol{x}^2(\omega^1),\omega^2,\boldsymbol{x}^3(\omega^1,\omega^2),\omega^3)\} \\
\text{subject to} \quad & \boldsymbol{T}^{2,1}(\omega^1,\omega^2)\boldsymbol{x}^1 + \boldsymbol{T}^{2,2}(\omega^1,\omega^2)\boldsymbol{x}^2(\omega^1) + \\
& \boldsymbol{T}^{2,3}(\omega^1,\omega^2)\boldsymbol{x}^3(\omega^1,\omega^2) = \boldsymbol{h}^2(\omega^1,\omega^2) \\
& \boldsymbol{x}^3(\omega^1,\omega^2) \in X^3 \Bigg\}, \ \forall \omega^1 \in \Omega^1, \forall \omega^2 \in \Omega^2,
\end{aligned}
$$

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$$
\begin{aligned}
Q^{r-1}(\boldsymbol{x}^1, \omega^1, \ldots, \omega^{r-1}) = \Bigg\{ & \\
\text{Minimize}_{\boldsymbol{x}^r(\omega^1,\ldots,\omega^{r-1})} \quad & \boldsymbol{c}^{r,\top}(\omega^1, \ldots, \omega^{r-1})\boldsymbol{x}^r(\omega^1, \ldots, \omega^{r-1}) \\
\text{subject to} \quad & \boldsymbol{T}^{r-1,1}(\omega^1, \ldots, \omega^{r-1})\boldsymbol{x}^1 + \\
& \boldsymbol{T}^{r-1,2}(\omega^1, \ldots, \omega^{r-1})\boldsymbol{x}^2(\omega^1) + \ldots + \\
& \boldsymbol{T}^{r-1,r}(\omega^1, \ldots, \omega^{r-1})\boldsymbol{x}^r(\omega^1, \ldots, \omega^{r-1}) = \\
& \boldsymbol{h}^{r-1}(\omega^1, \ldots, \omega^{r-1}) \\
& \boldsymbol{x}^r(\omega^1, \ldots, \omega^{r-1}) \in X^r \Bigg\}, \\
& \forall \omega^1 \in \Omega^1, \ldots, \forall \omega^{r-1} \in \Omega^{r-1}.
\end{aligned}
$$

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- Under rather general assumptions the multi-stage stochastic programming problem above can be equivalently expressed as the following deterministic problem:

$$
\begin{aligned}
&\text{Minimize}_{\boldsymbol{x}^1, \boldsymbol{x}^2(\omega^1), \ldots, \boldsymbol{x}^{\mathrm{r}}(\omega^1, \ldots, \omega^{\mathrm{r}-1})} \\
&\quad z = \\
&\quad \boldsymbol{c}^{1,\top} \boldsymbol{x}^1 + \\
&\quad \sum_{\omega^1 \in \Omega^1} \pi(\omega^1) \Big(\boldsymbol{c}^{2,\top}(\omega^1) \boldsymbol{x}^2(\omega^1) + \\
&\quad \sum_{\omega^2 \in \Omega^2} \pi(\omega^2) \big(\boldsymbol{c}^{3,\top}(\omega^1, \omega^2) \boldsymbol{x}^3(\omega^1, \omega^2) \big) + \ldots + \\
&\quad \sum_{\omega^{\mathrm{r}-1} \in \Omega^{\mathrm{r}-1}} \pi(\omega^{\mathrm{r}-1}) \big(\boldsymbol{c}^{\mathrm{r},\top}(\omega^1, \ldots, \omega^{\mathrm{r}-1}) \boldsymbol{x}^{\mathrm{r}}(\omega^1, \ldots, \omega^{\mathrm{r}-1}) \big) \Big)
\end{aligned}
$$

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subject to

$$
\begin{aligned}
&\boldsymbol{A}^1\boldsymbol{x}^1 = \boldsymbol{b}^1\\
&\boldsymbol{x}^1 \in X^1\\
&\boldsymbol{T}^{1,1}(\omega^1)\boldsymbol{x}^1 + \boldsymbol{T}^{1,2}(\omega^1)\boldsymbol{x}^2(\omega^1) = \boldsymbol{h}^1(\omega^1);\ \forall\omega^1 \in \Omega^1\\
&\boldsymbol{x}^2(\omega^1) \in X^2;\ \forall\omega^1 \in \Omega^1\\
&\boldsymbol{T}^{2,1}(\omega^1,\omega^2)\boldsymbol{x}^1 + \boldsymbol{T}^{2,2}(\omega^1,\omega^2)\boldsymbol{x}^2(\omega^1) +\\
&\boldsymbol{T}^{2,3}(\omega^1,\omega^2)\boldsymbol{x}^3(\omega^1,\omega^2) = \boldsymbol{h}^2(\omega^1,\omega^2);\ \forall\omega^1 \in \Omega^1, \forall\omega^2 \in \Omega^2\\
&\boldsymbol{x}^3(\omega^1,\omega^2) \in X^3,\ \forall\omega^1 \in \Omega^1, \forall\omega^2 \in \Omega^2\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots\\
&\boldsymbol{T}^{r-1,1}(\omega^1,\ldots,\omega^{r-1})\boldsymbol{x}^1 + \boldsymbol{T}^{r-1,2}(\omega^1,\ldots,\omega^{r-1})\boldsymbol{x}^2(\omega^1) + \ldots +\\
&\boldsymbol{T}^{r-1,r}(\omega^1,\ldots,\omega^{r-1})\boldsymbol{x}^r(\omega^1,\ldots,\omega^{r-1}) =\\
&\boldsymbol{h}^{r-1}(\omega^1,\ldots,\omega^{r-1});\ \forall\omega^1 \in \Omega^1,\ldots,\forall\omega^{r-1} \in \Omega^{r-1}\\
&\boldsymbol{x}^r(\omega^1,\ldots,\omega^{r-1}) \in X^r,\ \forall\omega^1 \in \Omega^1,\ldots,\forall\omega^{r-1} \in \Omega^{r-1}.
\end{aligned}
$$

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Example (Three-stage problem. Scenario-variable formulation)

Consider an electric energy producer with a production capacity of 120 MW. This producer needs to determine its selling strategy during a future market horizon of 6 days, divided into two 3-day periods.

The bilateral contracts available to this producer are the following:
1. A selling contract of up to 50 MW at $25/MWh at the beginning of the 6-day period and spanning the entire 6-day period.
2. A selling contract of up to 40 MW at $26/MWh just after the first 3-day period and covering the last 3-day period.

contract data for the producer

Contract #	Duration	Hours	Price ($/MWh)	Power Cap (MW)
A	Entire 6-day period	144	25	50
B	2nd 3-day period	72	26	40

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Complementarily to bilateral contracting, the producer can sell energy in the pool at constant power during the first and second 3-day periods.

Pool prices scenarios are described below:

1. Pool prices for the first 3-day period are \$27/MWh with probability 0.4 and \$23/MWh with probability 0.6.
2. If the price during the first 3-day period is \$27/MWh, prices during the second 3-day period are \$28/MWh with probability 0.4 and \$26/MWh with probability 0.6.
3. Alternatively, if the price during the first 3-day period is \$23/MWh, prices during the second 3-day period are \$24/MWh with probability 0.4 and \$22/MWh with probability 0.6.

scenario data for the producer

Scenario #	1st 3-day price (\$/MWh)	2nd 3-day price (\$/MWh)	Probability (per unit)
1	27	28	$0.4 \times 0.4 = 0.16$
2	27	26	$0.4 \times 0.6 = 0.24$
3	23	24	$0.6 \times 0.4 = 0.24$
4	23	22	$0.6 \times 0.6 = 0.36$

Stochastic Programming Fundamentals

- The three-stage decision-making problem is formulated as

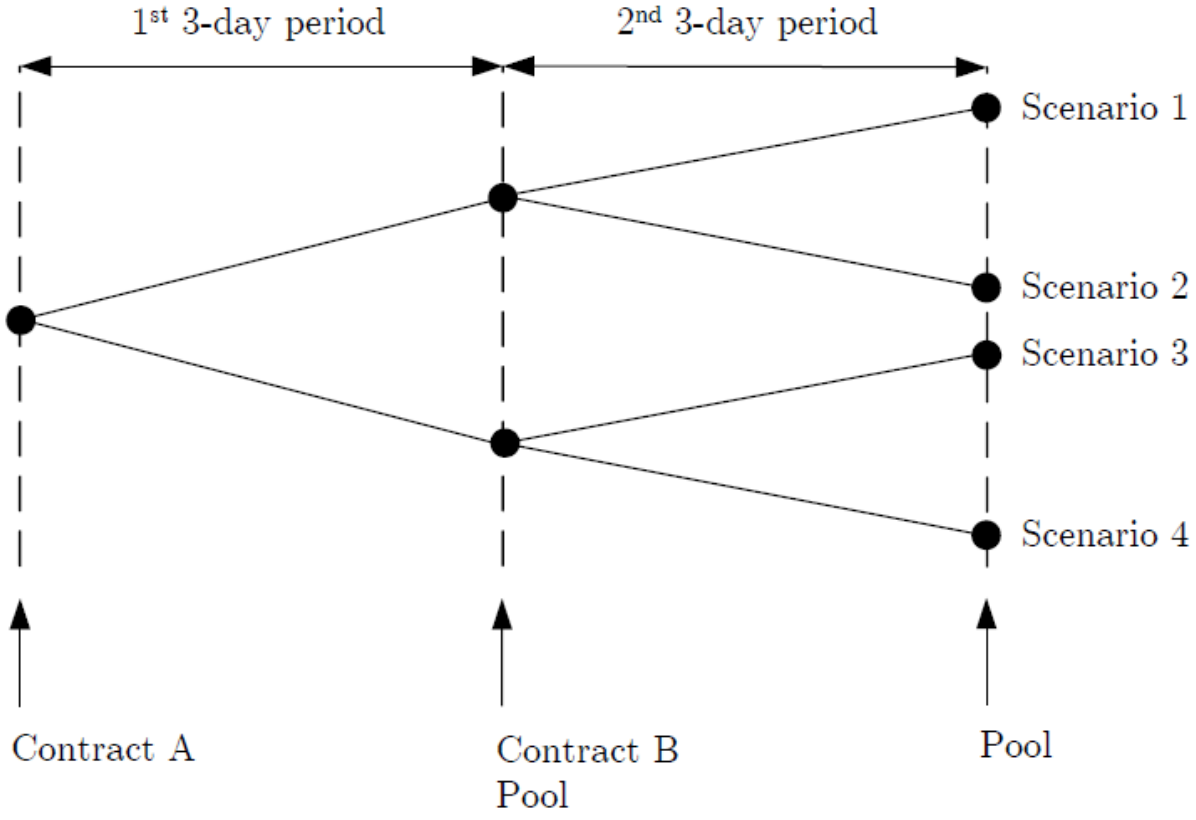
$$\text{Maximize}_{P_1^{\mathrm{A}},P_2^{\mathrm{A}},P_3^{\mathrm{A}},P_4^{\mathrm{A}},P_1^{\mathrm{B}},P_2^{\mathrm{B}},P_3^{\mathrm{B}},P_4^{\mathrm{B}},P_1^{t1},P_2^{t1},P_3^{t1},P_4^{t1},P_1^{t2},P_2^{t2},P_3^{t2},P_4^{t2}}$$

$$\Pi^{\mathrm{S}} =$$

$$0.16 \times 72(2 \times 25P_1^{\mathrm{A}} + 26P_1^{\mathrm{B}} + 27P_1^{t_1} + 28P_1^{t_2}) +$$
$$0.24 \times 72(2 \times 25P_2^{\mathrm{A}} + 26P_2^{\mathrm{B}} + 27P_2^{t_1} + 26P_2^{t_2}) +$$
$$0.24 \times 72(2 \times 25P_3^{\mathrm{A}} + 26P_3^{\mathrm{B}} + 23P_3^{t_1} + 24P_3^{t_2}) +$$
$$0.36 \times 72(2 \times 25P_4^{\mathrm{A}} + 26P_4^{\mathrm{B}} + 23P_4^{t_1} + 22P_4^{t_2})$$

subject to

$$P_1^{\mathrm{A}} + P_1^{t_1} \leq 120$$
$$P_2^{\mathrm{A}} + P_2^{t_1} \leq 120$$
$$P_3^{\mathrm{A}} + P_3^{t_1} \leq 120$$
$$P_4^{\mathrm{A}} + P_4^{t_1} \leq 120$$



Stochastic Programming Fundamentals

subject to

$$P_1^{\mathrm{A}} + P_1^{\mathrm{B}} + P_1^{t_2} \leq 120$$
$$P_2^{\mathrm{A}} + P_2^{\mathrm{B}} + P_2^{t_2} \leq 120$$
$$P_3^{\mathrm{A}} + P_3^{\mathrm{B}} + P_3^{t_2} \leq 120$$
$$P_4^{\mathrm{A}} + P_4^{\mathrm{B}} + P_4^{t_2} \leq 120$$

$$0 \leq P_1^{\mathrm{A}}, P_2^{\mathrm{A}}, P_3^{\mathrm{A}}, P_4^{\mathrm{A}} \leq 50$$
$$0 \leq P_1^{\mathrm{B}}, P_2^{\mathrm{B}}, P_3^{\mathrm{B}}, P_4^{\mathrm{B}} \leq 40$$
$$0 \leq P_1^{t_1}, P_2^{t_1}, P_3^{t_1}, P_4^{t_1}, P_1^{t_2}, P_2^{t_2}, P_3^{t_2}, P_4^{t_2}$$

$$P_1^{\mathrm{A}} = P_2^{\mathrm{A}} = P_3^{\mathrm{A}} = P_4^{\mathrm{A}}$$
$$P_1^{\mathrm{B}} = P_2^{\mathrm{B}}, P_3^{\mathrm{B}} = P_4^{\mathrm{B}}$$
$$P_1^{t_1} = P_2^{t_1}, P_3^{t_1} = P_4^{t_1}$$

Variable Π^{S} is the expected profit. Superscript S stands for stochastic.
Variables $P_1^{\mathrm{A}}, P_2^{\mathrm{A}}, P_3^{\mathrm{A}}$, and P_4^{A} represent the power to be sold through the contract spanning the whole 6-day period, and correspond to scenarios $1, 2, 3$, and 4, respectively.
Variables $P_1^{\mathrm{B}}, P_2^{\mathrm{B}}, P_3^{\mathrm{B}}$, and P_4^{B} represent the power to be sold through the contract spanning the second 3-day period, and correspond to scenarios $1, 2, 3$, and 4, respectively.
Variables $P_1^{t_1}, P_2^{t_1}, P_3^{t_1}$, and $P_4^{t_1}$ represent the power to be sold in the pool that spans the first 3-day period, and correspond to scenarios $1, 2, 3$, and 4, respectively.
Variables $P_1^{t_2}, P_2^{t_2}, P_3^{t_2}$, and $P_4^{t_2}$ represent the power to be sold in the pool during the second 3-day period, and correspond to scenarios $1, 2, 3$, and 4, respectively.

1. $P_1^{\mathrm{A}*} = P_2^{\mathrm{A}*} = P_3^{\mathrm{A}*} = P_4^{\mathrm{A}*} = 50$
2. $P_1^{\mathrm{B}*} = P_2^{\mathrm{B}*} = 0,\ P_3^{\mathrm{B}*} = P_4^{\mathrm{B}*} = 40$
3. $P_1^{t_1*} = P_2^{t_1*} = 70,\ P_3^{t_1*} = P_4^{t_1*} = 70$
4. $P_1^{t_2*} = 70,\ P_2^{t_2*} = 70,\ P_3^{t_2*} = 30,\ P_4^{t_2*} = 30.$

Stochastic Programming Fundamentals

Example (Three-stage problem. Node-variable formulation of previous example)

$$\text{Maximize}_{P^{\mathrm{A}},P_1^{\mathrm{B}},P_3^{\mathrm{B}},P_1^{t_1},P_3^{t_1},P_1^{t_2},P_2^{t_2},P_3^{t_2},P_4^{t_2}}$$

$$\Pi^{\mathrm{S}} = 25 \times 144P^{\mathrm{A}} +$$

$$0.4 \times 72(26P_1^{\mathrm{B}} + 27P_1^{t_1} + 0.4 \times 28P_1^{t_2} + 0.6 \times 26P_2^{t_2}) +$$

$$0.6 \times 72(26P_3^{\mathrm{B}} + 23P_3^{t_1} + 0.4 \times 24P_3^{t_2} + 0.6 \times 22P_4^{t_2})$$

subject to

$$P^{\mathrm{A}} + P_1^{t_1} \leq 120$$

$$P^{\mathrm{A}} + P_3^{t_1} \leq 120$$

$$P^{\mathrm{A}} + P_1^{\mathrm{B}} + P_1^{t_2} \leq 120$$

$$P^{\mathrm{A}} + P_1^{\mathrm{B}} + P_2^{t_2} \leq 120$$

$$P^{\mathrm{A}} + P_3^{\mathrm{B}} + P_3^{t_2} \leq 120$$

$$P^{\mathrm{A}} + P_3^{\mathrm{B}} + P_4^{t_2} \leq 120$$

$$0 \leq P^{\mathrm{A}} \leq 50$$

$$0 \leq P_1^{\mathrm{B}}, P_3^{\mathrm{B}} \leq 40$$

$$0 \leq P_1^{t_1}, P_3^{t_1}, P_1^{t_2}, P_2^{t_2}, P_3^{t_2}, P_4^{t_2} \ .$$

Stochastic Programming Fundamentals

❑ Quality Metrics

- It is always a good practice to justify adequately why stochastic programming problems are solved instead of deterministic ones.
- It is also important to assess how relevant is to have accurate forecasting procedures to identify the most likely scenarios.
- Two metrics are usually used to appraise the interest of using stochastic programming models: the Expected Value of Perfect Information (EVPI) and the Value of the Stochastic Solution (VSS).
- Additionally, out-of-sample techniques should be used to asses the quality of the outcomes provided by the solutions obtained using stochastic programming models.

Stochastic Programming Fundamentals

❑ Expected Value of Perfect Information

Assuming a maximization two-stage stochastic programming problem, denoted by z^{S*} its objective function optimal value, which represents the expected value of profit over all scenarios. Superscript *S* stands for stochastic.

On the other hand, z^{P*} is the objective function optimal value of the two-stage stochastic programming problem in which non-anticipativity constraints are relaxed (a scenario-variable formulation is assumed).
Superscript *P* stands for perfect information.

Stochastic Programming Fundamentals

The expected value of perfect information (EVPI) is then computed as

$$\mathrm{EVPI}_{\max} = z^{\mathrm{P}*} - z^{\mathrm{S}*}.$$

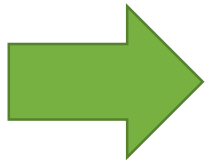
Note that for minimization problems EVPI is given by

$$\mathrm{EVPI}_{\min} = z^{\mathrm{S}*} - z^{\mathrm{P}*}.$$

Stochastic Programming Fundamentals

Example (EVPI for a two-stage problem)

$$
\begin{aligned}
&\text{Minimize}_{P_1^{\mathrm{C}},P_2^{\mathrm{C}},P_3^{\mathrm{C}},P_1,P_2,P_3} \\
&C^{\mathrm{P}} = 0.2 \times 168(45P_1^{\mathrm{C}} + 50P_1) \\
&\qquad + 0.6 \times 168(45P_2^{\mathrm{C}} + 46P_2) + 0.2 \times 168(45P_3^{\mathrm{C}} + 44P_3) \\
&\text{subject to} \\
&P_1^{\mathrm{C}} + P_1 \geq 110 \\
&P_2^{\mathrm{C}} + P_2 \geq 100 \\
&P_3^{\mathrm{C}} + P_3 \geq 80 \\
&0 \leq P_1^{\mathrm{C}}, P_2^{\mathrm{C}}, P_3^{\mathrm{C}} \leq 90 \\
&0 \leq P_1, P_2, P_3 \ ,
\end{aligned}
$$



$$
\begin{aligned}
C^{\mathrm{P}*} &= 742{,}560, \\
P_1^{\mathrm{C}*} &= 90, \\
P_2^{\mathrm{C}*} &= 90, \\
P_3^{\mathrm{C}*} &= 0, \\
P_1^{*} &= 20, \\
P_2^{*} &= 10, \\
P_3^{*} &= 80.
\end{aligned}
$$

Stochastic Programming Fundamentals

The value of EVPI is thus

$$\text{EVPI} = C^{\text{S}*} - C^{\text{P}*} = 747{,}936 - 742{,}560 = \$5376$$

where $C^{\text{S}*}$ is the objective function optimal value obtained from the solution of the stochastic programming problem.

Stochastic Programming Fundamentals

Example (EVPI for a three-stage problem)

$$\text{Maximize}_{P_1^{\text{A}},P_2^{\text{A}},P_3^{\text{A}},P_4^{\text{A}},P_1^{\text{B}},P_2^{\text{B}},P_3^{\text{B}},P_4^{\text{B}},P_1^{t1},P_2^{t1},P_3^{t1},P_4^{t1},P_1^{t2},P_2^{t2},P_3^{t2},P_4^{t2}}$$

$$\Pi^{\text{P}_1} =$$

$$0.16 \times 72(2 \times 25P_1^{\text{A}} + 26P_1^{\text{B}} + 27P_1^{t_1} + 28P_1^{t_2}) +$$

$$0.24 \times 72(2 \times 25P_2^{\text{A}} + 26P_2^{\text{B}} + 27P_2^{t_1} + 26P_2^{t_2}) +$$

$$0.24 \times 72(2 \times 25P_3^{\text{A}} + 26P_3^{\text{B}} + 23P_3^{t_1} + 24P_3^{t_2}) +$$

$$0.36 \times 72(2 \times 25P_4^{\text{A}} + 26P_4^{\text{B}} + 23P_4^{t_1} + 22P_4^{t_2})$$

Stochastic Programming Fundamentals

subject to

$$
\begin{aligned}
&P_1^{\mathrm{A}} + P_1^{t_1} \leq 120 \\
&P_2^{\mathrm{A}} + P_2^{t_1} \leq 120 \\
&P_3^{\mathrm{A}} + P_3^{t_1} \leq 120 \\
&P_4^{\mathrm{A}} + P_4^{t_1} \leq 120 \\
&P_1^{\mathrm{A}} + P_1^{\mathrm{B}} + P_1^{t_2} \leq 120 \\
&P_2^{\mathrm{A}} + P_2^{\mathrm{B}} + P_2^{t_2} \leq 120 \\
&P_3^{\mathrm{A}} + P_3^{\mathrm{B}} + P_3^{t_2} \leq 120 \\
&P_4^{\mathrm{A}} + P_4^{\mathrm{B}} + P_4^{t_2} \leq 120 \\
&0 \leq P_1^{\mathrm{A}}, P_2^{\mathrm{A}}, P_3^{\mathrm{A}}, P_4^{\mathrm{A}} \leq 50 \\
&0 \leq P_1^{\mathrm{B}}, P_2^{\mathrm{B}}, P_3^{\mathrm{B}}, P_4^{\mathrm{B}} \leq 40 \\
&0 \leq P_1^{t_1}, P_2^{t_1}, P_3^{t_1}, P_4^{t_1}, P_1^{t_2}, P_2^{t_2}, P_3^{t_2}, P_4^{t_2} \\
&P_1^{\mathrm{A}} = P_2^{\mathrm{A}}, P_3^{\mathrm{A}} = P_4^{\mathrm{A}} \\
&P_1^{\mathrm{B}} = P_2^{\mathrm{B}}, P_3^{\mathrm{B}} = P_4^{\mathrm{B}} \\
&P_1^{t_1} = P_2^{t_1}, P_3^{t_1} = P_4^{t_1} \ .
\end{aligned}
$$

- ✓ Note that the non-anticipativity constraints have been modified to enforce that there is no uncertainty in the first stage.

$$\Pi^{\mathrm{P1}*} = \$437{,}961.6$$

Stochastic Programming Fundamentals

- If we further assume that there is no uncertainty in both the first and second stages, then the perfect information problem to be solved becomes

$$\text{Maximize}_{P_1^{\mathrm{A}},P_2^{\mathrm{A}},P_3^{\mathrm{A}},P_4^{\mathrm{A}},P_1^{\mathrm{B}},P_2^{\mathrm{B}},P_3^{\mathrm{B}},P_4^{\mathrm{B}},P_1^{t_1},P_2^{t_1},P_3^{t_1},P_4^{t_1},P_1^{t_2},P_2^{t_2},P_3^{t_2},P_4^{t_2}}$$

$$\Pi^{\mathrm{P}_2} =$$

$$0.16 \times 72(2 \times 25P_1^{\mathrm{A}} + 26P_1^{\mathrm{B}} + 27P_1^{t_1} + 28P_1^{t_2}) +$$
$$0.24 \times 72(2 \times 25P_2^{\mathrm{A}} + 26P_2^{\mathrm{B}} + 27P_2^{t_1} + 26P_2^{t_2}) +$$
$$0.24 \times 72(2 \times 25P_3^{\mathrm{A}} + 26P_3^{\mathrm{B}} + 23P_3^{t_1} + 24P_3^{t_2}) +$$
$$0.36 \times 72(2 \times 25P_4^{\mathrm{A}} + 26P_4^{\mathrm{B}} + 23P_4^{t_1} + 22P_4^{t_2})$$

subject to

$$P_1^{\mathrm{A}} + P_1^{t_1} \leq 120$$
$$P_2^{\mathrm{A}} + P_2^{t_1} \leq 120$$
$$P_3^{\mathrm{A}} + P_3^{t_1} \leq 120$$
$$P_4^{\mathrm{A}} + P_4^{t_1} \leq 120$$

$$P_1^{\mathrm{A}} + P_1^{\mathrm{B}} + P_1^{t_2} \leq 120$$
$$P_2^{\mathrm{A}} + P_2^{\mathrm{B}} + P_2^{t_2} \leq 120$$
$$P_3^{\mathrm{A}} + P_3^{\mathrm{B}} + P_3^{t_2} \leq 120$$
$$P_4^{\mathrm{A}} + P_4^{\mathrm{B}} + P_4^{t_2} \leq 120$$
$$0 \leq P_1^{\mathrm{A}}, P_2^{\mathrm{A}}, P_3^{\mathrm{A}}, P_4^{\mathrm{A}} \leq 50$$
$$0 \leq P_1^{\mathrm{B}}, P_2^{\mathrm{B}}, P_3^{\mathrm{B}}, P_4^{\mathrm{B}} \leq 40$$
$$0 \leq P_1^{t_1}, P_2^{t_1}, P_3^{t_1}, P_4^{t_1}, P_1^{t_2}, P_2^{t_2}, P_3^{t_2}, P_4^{t_2} \ .$$

Stochastic Programming Fundamentals

$$\Pi^{\mathrm{P2}*} = \$437{,}961.6$$

$$\Pi^{\mathrm{P2}*} \geq \Pi^{\mathrm{P1}*} \geq \Pi^{\mathrm{S}*}, \ \Pi^{\mathrm{S}*} = \$432{,}489.6$$

$$\mathrm{EVPI}_1 = \Pi^{\mathrm{P2}*} - \Pi^{\mathrm{P1}*} = 437{,}961.6 - 437{,}961.6 = 0$$
$$\mathrm{EVPI}_2 = \Pi^{\mathrm{P1}*} - \Pi^{\mathrm{S}*} = 437{,}961.6 - 432{,}489.6 = \$5472.$$

$$\mathrm{EVPI} = \mathrm{EVPI}_1 + \mathrm{EVPI}_2 = \$5472$$

Stochastic Programming Fundamentals

❑ Value of the Stochastic Solution

- The value of the stochastic solution is a measure to quantify the advantage of using a stochastic programming approach over a deterministic one.
- The original stochastic programming problem can then be solved fixing the values of the first-stage variables to those provided by the deterministic one. This modified problem decomposes by scenario and is generally easy to solve.
- The optimal objective function value of the modified stochastic problem (with fixed first stage
- decisions) is denoted by $z^{\mathrm{D}*}$. Superscript D stands for deterministic.

Stochastic Programming Fundamentals

The value of the stochastic solution (VSS) is then calculated as

$$\mathrm{VSS}_{\max} = z^{\mathrm{S}*} - z^{\mathrm{D}*}$$

For minimization problems, the VSS becomes

$$\mathrm{VSS}_{\min} = z^{\mathrm{D}*} - z^{\mathrm{S}*}$$

Stochastic Programming Fundamentals

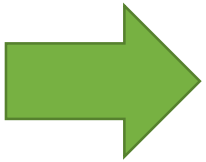
Example (VSS for a two-stage problem)

Consider the consumer problem stated in Illustrative Example (Two-stage problem. Node-variable formulation). The average scenario is characterized as

$$0.2 \times 110 + 0.6 \times 100 + 0.2 \times 80 = 98 \text{ MWh}$$
$$0.2 \times 50 + 0.6 \times 46 + 0.2 \times 44 = \$46.4/\text{MWh}$$

The problem related to this average scenario is

$$\begin{aligned}
&\text{Minimize}_{P^{\text{C}},P} \\
&168(45P^{\text{C}} + 46.4P) \\
&\text{subject to} \\
&P^{\text{C}} + P \geq 98 \\
&0 \leq P^{\text{C}} \leq 90 \\
&0 \leq P\,,
\end{aligned}$$

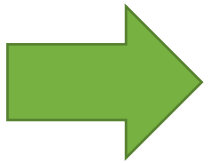


$$P_{\text{D}}^{\text{C}} = 90 \text{ MW}$$

Stochastic Programming Fundamentals

The subsequent multi-scenario problem to be solved with fixed first stage decision is

$$\text{Minimize}_{P_1,P_2,P_3}$$
$$C^{\mathrm{D}} = 168(45P_{\mathrm{D}}^{\mathrm{C}} + 0.2 \times 50P_1 + 0.6 \times 46P_2 + 0.2 \times 44P_3)$$
$$\text{subject to}$$
$$P_1 \geq 110 - P_{\mathrm{D}}^{\mathrm{C}}$$
$$P_2 \geq 100 - P_{\mathrm{D}}^{\mathrm{C}}$$
$$P_3 \geq 80 - P_{\mathrm{D}}^{\mathrm{C}}$$
$$0 \leq P_1, P_2, P_3 \ .$$



$$P_1^* = 20$$
$$P_2^* = 10$$
$$P_3^* = 0$$

- ✓ Which means that the consumer buys during the week 20, 10 and 0 MW for demands 110, 100 and 80 MW, respectively.

Stochastic Programming Fundamentals

Note that the problem above decomposes by scenario and renders the trivial problems below:

$$\text{Minimize}_{P_1}$$
$$C_1^{\mathrm{D}} = 168(0.2 \times 50P_1)$$
$$\text{subject to}$$
$$P_1 \geq 110 - P_{\mathrm{D}}^{\mathrm{C}}$$
$$0 \leq P_1,$$
$$\text{Minimize}_{P_2}$$
$$C_2^{\mathrm{D}} = 168(0.6 \times 46P_2)$$
$$\text{subject to}$$
$$P_2 \geq 100 - P_{\mathrm{D}}^{\mathrm{C}}$$
$$0 \leq P_2,$$
$$\text{Minimize}_{P_3}$$
$$C_3^{\mathrm{D}} = 168(0.2 \times 44P_3)$$
$$\text{subject to}$$
$$P_3 \geq 80 - P_{\mathrm{D}}^{\mathrm{C}}$$
$$0 \leq P_3,$$

AND $C^{\mathrm{D}} = 168(45P_{\mathrm{D}}^{\mathrm{C}}) + C_1^{\mathrm{D}} + C_2^{\mathrm{D}} + C_3^{\mathrm{D}}.$

$$\mathrm{VSS} = C^{\mathrm{D}*} - C^{\mathrm{S}*} = 760{,}368 - 747{,}936 = \$12{,}432\ .$$

Stochastic Programming Fundamentals

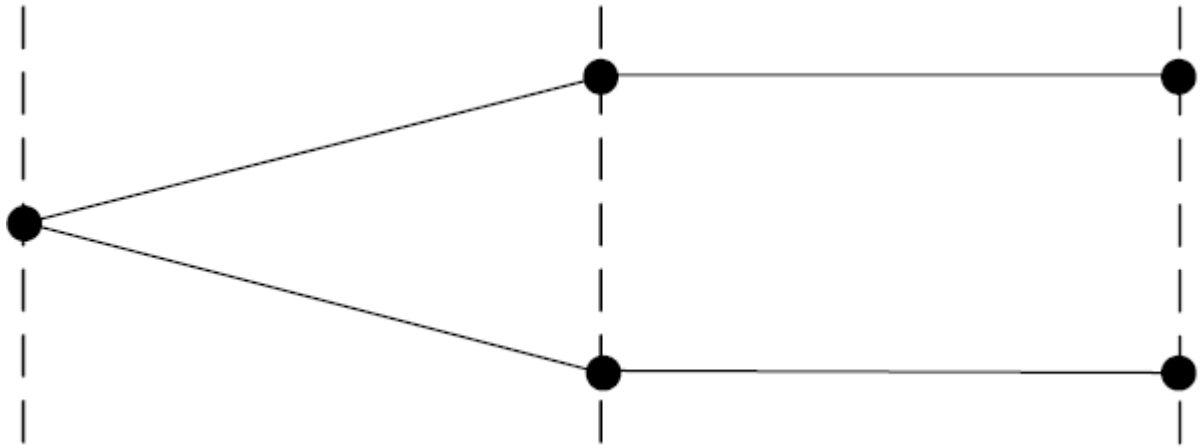
Example (VSS for a three-stage problem)

- If the price during the first 3-day period is \$27/MWh, the average price for the second 3-day period is

$$0.4 \times 28 + 0.6 \times 26 = \$26.8/\text{MWh}.$$

- On the other hand, if the price during the first 3-day period is \$23/MWh, the average price for the second 3-day period is

$$0.4 \times 24 + 0.6 \times 22 = \$22.8/\text{MWh}.$$



Stochastic Programming Fundamentals

Taking into account these average prices, the decision-making problem becomes

$$
\begin{aligned}
&\text{Maximize}_{P^{\mathrm{A}},P_1^{\mathrm{B}},P_3^{\mathrm{B}},P_1^{t_1},P_3^{t_1},P_1^{t_2},P_2^{t_2}} \\
&25 \times 144P^{\mathrm{A}} + \\
&0.4 \times 72(26P_1^{\mathrm{B}} + 27P_1^{t_1} + 26.8P_1^{t_2}) + \\
&0.6 \times 72(26P_3^{\mathrm{B}} + 23P_3^{t_1} + 22.8P_2^{t_2}) \\
&\text{subject to} \\
&P^{\mathrm{A}} + P_1^{t_1} \leq 120 \\
&P^{\mathrm{A}} + P_3^{t_1} \leq 120 \\
&P^{\mathrm{A}} + P_1^{\mathrm{B}} + P_1^{t_2} \leq 120 \\
&P^{\mathrm{A}} + P_3^{\mathrm{B}} + P_2^{t_2} \leq 120 \\
&0 \leq P^{\mathrm{A}} \leq 50 \\
&0 \leq P_1^{\mathrm{B}}, P_3^{\mathrm{B}} \leq 40 \\
&0 \leq P_1^{t_1}, P_3^{t_1}, P_1^{t_2}, P_2^{t_2} \ .
\end{aligned}
$$



$$P_{\mathrm{D1}}^{\mathrm{A}} = 50 \text{ MW}$$

Stochastic Programming Fundamentals

This value is fixed in the multi-scenario problem below:

$$\text{Maximize}_{P_1^{\mathrm{B}},P_3^{\mathrm{B}},P_1^{t_1},P_3^{t_1},P_1^{t_2},P_2^{t_2},P_3^{t_2},P_4^{t_2}}$$

$$\Pi^{\mathrm{D1}} = 25 \times 144 P_{\mathrm{D1}}^{\mathrm{A}} +$$

$$0.4 \times 72(26P_1^{\mathrm{B}} + 27P_1^{t_1} + 0.4 \times 28P_1^{t_2} + 0.6 \times 26P_2^{t_2}) +$$

$$0.6 \times 72(26P_3^{\mathrm{B}} + 23P_3^{t_1} + 0.4 \times 24P_3^{t_2} + 0.6 \times 22P_4^{t_2})$$

subject to

$$P_1^{t_1} \leq 120 - P_{\mathrm{D1}}^{\mathrm{A}}$$

$$P_3^{t_1} \leq 120 - P_{\mathrm{D1}}^{\mathrm{A}}$$

$$P_1^{\mathrm{B}} + P_1^{t_2} \leq 120 - P_{\mathrm{D1}}^{\mathrm{A}}$$

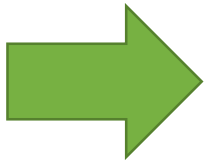
$$P_1^{\mathrm{B}} + P_2^{t_2} \leq 120 - P_{\mathrm{D1}}^{\mathrm{A}}$$

$$P_3^{\mathrm{B}} + P_3^{t_2} \leq 120 - P_{\mathrm{D1}}^{\mathrm{A}}$$

$$P_3^{\mathrm{B}} + P_4^{t_2} \leq 120 - P_{\mathrm{D1}}^{\mathrm{A}}$$

$$0 \leq P_1^{\mathrm{B}}, P_3^{\mathrm{B}} \leq 40$$

$$0 \leq P_1^{t_1}, P_3^{t_1}, P_1^{t_2}, P_2^{t_2}, P_3^{t_2}, P_4^{t_2} \ .$$



$$\Pi^{\mathrm{D1}*} = \$432{,}489.6$$

Stochastic Programming Fundamentals

Next, let us consider the average prices in both the first and second stages,

$0.4 \times 27 + 0.6 \times 23 = \$24.6/\text{MWh}$

$0.16 \times 28 + 0.24 \times 26 + 0.24 \times 24 + 0.36 \times 22 = \$24.4/\text{MWh}.$

✓ For these average prices, the decision-making problem becomes

$$\text{Maximize}_{P^{\text{A}}, P^{\text{B}}, P^{t_1}, P^{t_2}}$$

$$25 \times 144 P^{\text{A}} + 72(26 P^{\text{B}} + 24.6 P^{t_1} + 24.4 P^{t_2})$$

$$\text{subject to}$$

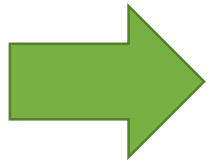
$$P^{\text{A}} + P^{t_1} \leq 120$$

$$P^{\text{A}} + P^{\text{B}} + P^{t_2} \leq 120$$

$$0 \leq P^{\text{A}} \leq 50$$

$$0 \leq P^{\text{B}} \leq 40$$

$$0 \leq P^{t_1}, P^{t_2} .$$



$$P^{\text{A}}_{\text{D2}} = 50$$

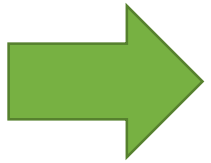
$$P^{\text{B}}_{\text{D2}} = 40$$

$$P^{t_1}_{\text{D2}} = 70$$

Stochastic Programming Fundamentals

These values are fixed in the multi-scenario problem below:

$$
\begin{aligned}
&\text{Maximize}_{P_1^{t2},P_2^{t2},P_3^{t2},P_4^{t2}} \\
&\Pi^{\mathrm{D2}} = 25 \times 144 P_{\mathrm{D2}}^{\mathrm{A}} + \\
&0.4 \times 72(26 P_{\mathrm{D2}}^{B} + 27 P_{\mathrm{D2}}^{t1} + 0.4 \times 28 P_1^{t2} + 0.6 \times 26 P_2^{t2}) + \\
&0.6 \times 72(26 P_{\mathrm{D2}}^{\mathrm{B}} + 23 P_{\mathrm{D2}}^{t1} + 0.4 \times 24 P_3^{t2} + 0.6 \times 22 P_4^{t2}) \\
&\text{subject to} \\
&P_1^{t2} \le 120 - P_{\mathrm{D2}}^{\mathrm{A}} - P_{\mathrm{D2}}^{B} \\
&P_2^{t2} \le 120 - P_{\mathrm{D2}}^{\mathrm{A}} - P_{\mathrm{D2}}^{B} \\
&P_3^{t2} \le 120 - P_{\mathrm{D2}}^{\mathrm{A}} - P_{\mathrm{D2}}^{B} \\
&P_4^{t2} \le 120 - P_{\mathrm{D2}}^{\mathrm{A}} - P_{\mathrm{D2}}^{B} \\
&0 \le P_1^{t2}, P_2^{t2}, P_3^{t2}, P_4^{t2} \ .
\end{aligned}
$$



$$\Pi^{\mathrm{D2}*} = \$431{,}568$$

Stochastic Programming Fundamentals

Given the maximization nature of the problem considered, it follows that

$$\Pi^{S*} \geq \Pi^{D1*} \geq \Pi^{D2*} , \quad \Pi^{S*} = \$432{,}489.6$$

$$\mathrm{VSS}_1 = \Pi^{D1} - \Pi^{D2} = 432{,}489.6 - 431{,}568 = \$921.6$$
$$\mathrm{VSS}_2 = \Pi^{S} - \Pi^{D1} = 432{,}489.6 - 432{,}489.6 = 0$$

$$\mathrm{VSS} = \mathrm{VSS}_1 + \mathrm{VSS}_2 = \$921.6 .$$

Stochastic Programming Fundamentals

❑ Out-of-Sample Assessment

- The purpose of an out-of-sample assessment is to evaluate the effectiveness of the decisions obtained by solving a stochastic programming problem in terms of actual real-world outcomes.
- Additionally, out-of-sample simulations can be used to compare the effectiveness in terms of actual outcomes of alternative decisions, e.g., the decision obtained by solving a stochastic programming model and that obtained via a simplified deterministic one.

Stochastic Programming Fundamentals

- Roughly speaking, an out-of-sample simulation works as follows:

1. Solve a stochastic programming model to obtain the decision to be made at time t with all the available information up to time $t-1$. For instance, a producer derives offering curves for the day-ahead market (of certain electricity market) pertaining to day d with all the data available prior to the clearing of that market. For that, it uses a stochastic programming model.
2. Implement the decision obtained by solving the stochastic programming problem of item 1 above, and observe the outcome obtained that corresponds to time t. For instance, the producer does offer in the market the curves obtained in step 1 above and observes the actual profit it obtains.
3. Steps 1 and 2 above are repeated for times $t = 1, \ldots, T$ to statistically characterize the resulting outcome. T should be large enough. This way the producer in the considered example characterizes its profit.

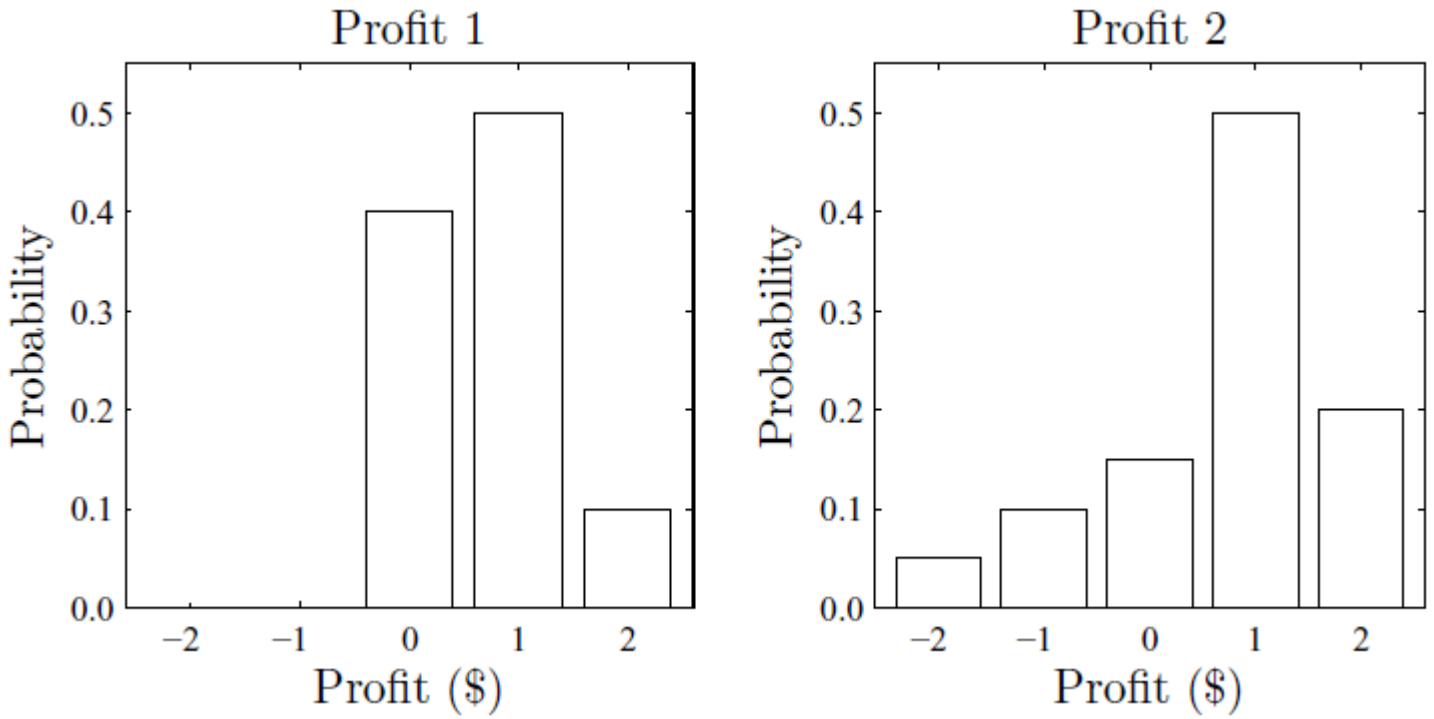
✓ **Since out-of-sample techniques allow characterizing the consequences of decisions in terms of outcome distributions, they provide a most effective way to evaluate the quality of a decision and of the decision tool used to generate it.**

Stochastic Programming Fundamentals

❑ Risk

- Typically, stochastic programming problems maximize (minimize) an objective function representing an expected profit (cost). These problems constitute risk-neutral models.
- If risk is accounted for, the decision maker also assesses the profit values in the worst scenarios, in addition to the expected value of the profit. In this case, the decision maker becomes a risk-averse agent.
- Since the risk-aversion level is an input parameter for all the considered models, a decision maker should know in advance the level of risk he/she is willing to assume.

Stochastic Programming Fundamentals



This figure shows the probability mass functions of two profit variables denoted by Profit 1 and Profit 2. The expected value of both profit variables is the same, $0.7. Therefore, both profits are equally good for a risk-neutral decision maker. However, note that Profit 1 is always positive, i.e., the decision maker does not experience loss in any scenario. On the other hand, the probability of having a negative profit is 0.15 in Profit 2. Hence, we may conclude that Profit 2 is riskier than Profit 1.

Stochastic Programming Fundamentals

❑ Solving Stochastic Programming Problems

Since stochastic programming problems generally include complicating constraints, i.e., constraints that if relaxed render the resulting problem easy to solve as it decomposes by scenario. Therefore, decomposition techniques are particularly suited to tackle stochastic programming problems.

Some general rules on which solution technique to be used follow:

1. If the problem under consideration is linear and involves no discrete variables, its solution can be addressed directly for sizes up to a few millions variables/constrains.
2. If the problem is linear but involves discrete variables, depending on the number of discrete variables and the structure of the problem, either a direct solution technique or a decomposition procedure would be advisable.
3. Similarly, if the problem is nonlinear and continuous, depending on the number of variables and the structure of the problem, either a direct solution technique or a decomposition procedure would be advisable.
4. Finally, if the problem is nonlinear and involves continuous as well as discrete variables, a decomposition procedure is generally advisable.