Monte Carlo Simulation

Behnam Mohammadi-Ivatloo

Monte Carlo Simulation

❖ Monte Carlo Simulation

- A numerical simulation is a process of selecting a set of values of system parameters and obtaining a solution of the system model for a selected set.
- By repeating the simulation process for different sets of system parameters, we obtain different sample solutions.
- The key activity in the simulation process is the selection of system parameters to obtain sample solutions.
- A numerical simulation procedure applied to problems involving random variables with known (or assumed) probability distributions is called **Monte Carlo simulation**.

Monte Carlo Simulation

❖ Monte Carlo Simulation

- Monte Carlo simulation involves repeating the simulation process, using in each simulation a particular set of values of the random variables generated in accordance with the corresponding probability distributions.

- A sample from a Monte Carlo simulation is similar to a sample of experimental observations. Therefore, the results of Monte Carlo simulations may be treated statistically, and the methods of statistical estimation and inference are applicable.

- The essence of Monte Carlo simulation may be described with a specific example. For this purpose, consider the generation reliability problem discussed in **Example 7.2**. Our aim is to obtain an estimate of the **loss of load probability (LOLP),** defined as:

$$LOLP = P(G - L < 0)$$

Random variable representing the available generation

Random variable representing the available load

Monte Carlo Simulation

❖ Monte Carlo Simulation

- To obtain an estimate of the LOLP we will need to generate sample pairs of values of variables G and L and check whether the generation is smaller than the load for each pair.

- The sample values of G and L are selected in accordance with their probability distributions as follows.

- A value of the load variable is selected from the probability distribution of the daily peak load in **Fig. 13.l**.

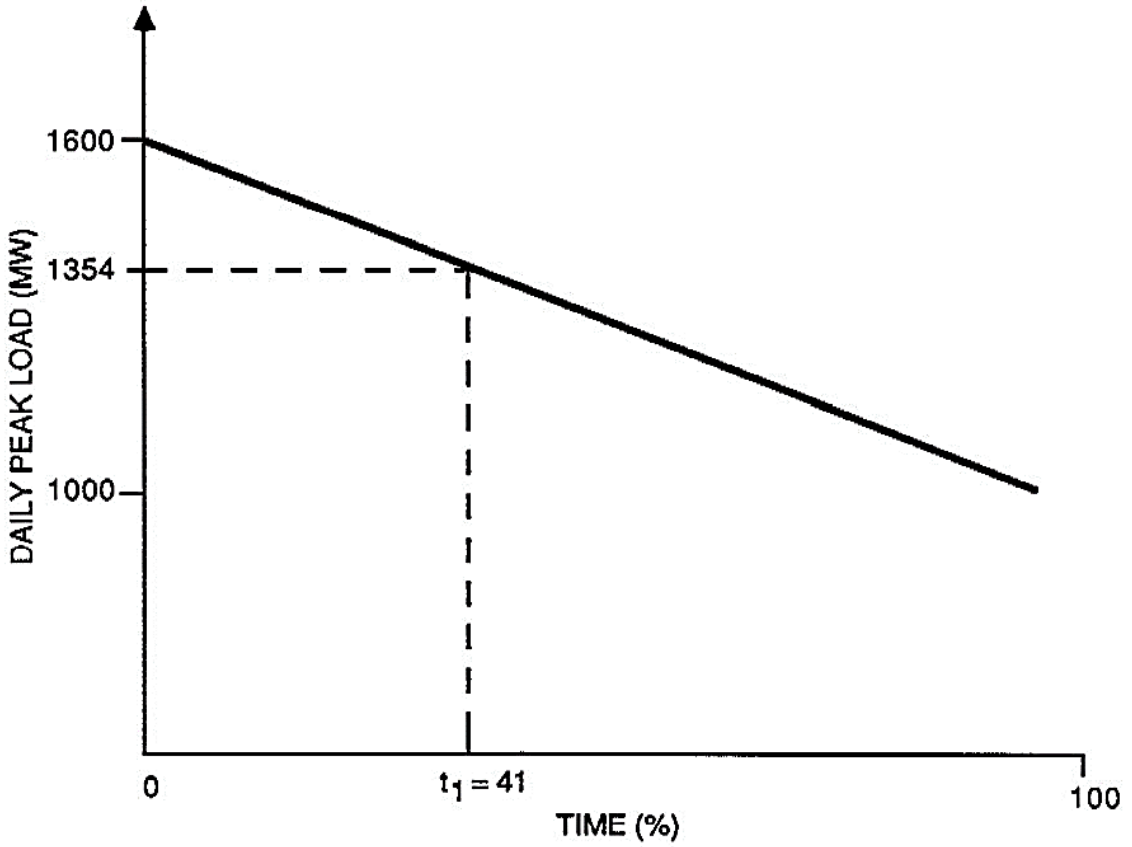


Figure 13.1 Selection of a load value in Monte Carlo simulation.

❖ Monte Carlo Simulation

- To select a value of available generation we first construct a cumulative distribution function of G.

- In this **example**, there are four 550-MW generating units, each with a probability of failure of 0.264. The available generating capacity is a random variable with a binomial distribution having parameter p = 0.264. The probability distribution function is shown in Fig. 13.2.

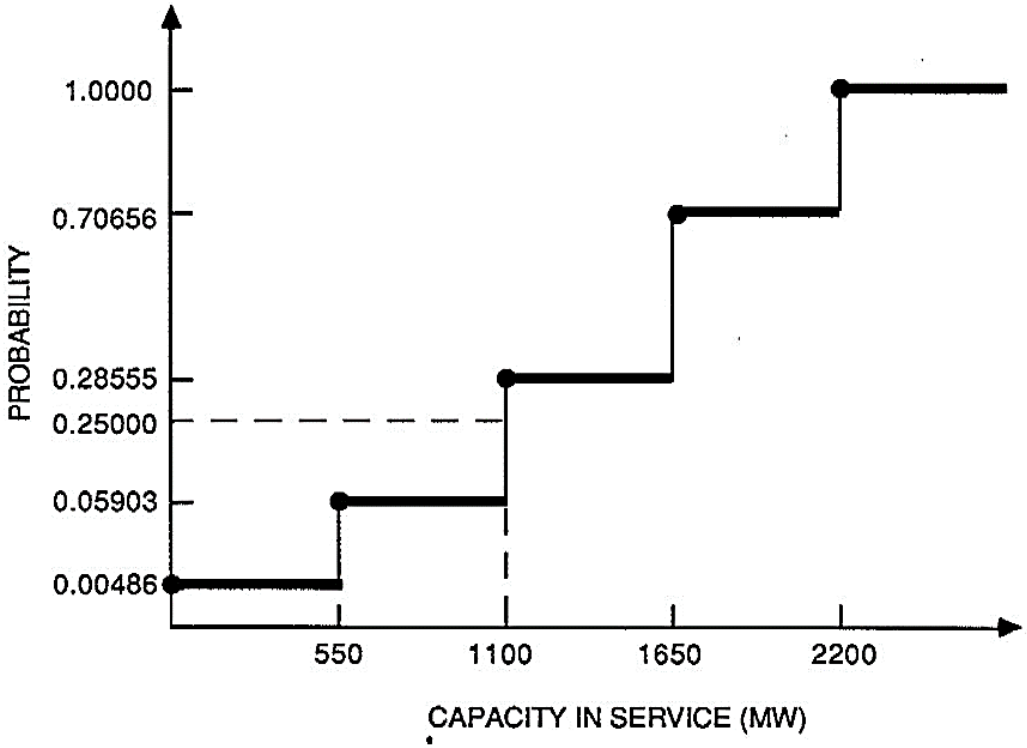


Figure 13.2 Probability distribution function of available generating capacity.

Monte Carlo Simulation

❖ Monte Carlo Simulation

- Suppose that we have selected randomly a number between O and 1. The corresponding available capacity is read from Fig. 13.2 and is equal to 1100 MW in this case.

- Since the available generation is smaller than load, this Monte Carlo sample, shown as sample No. 1 in Table 13.1, represents the system failure condition.

- Another pair of generation and load values is similarly generated as shown for sample No. 2 in Table 13.1; in this case, there is sufficient generation to meet the load.

TABLE 13.1 Ten Samples of Monte Carlo Simulation for LOLP Computations

Sample No.	Load (MW)	Generation (MW)	Status[a]
1	1354	1100	F
2	1045	2200	S
3	1459	1100	F
4	1187	1650	S
5	1654	550	F
6	1516	1650	S
7	1272	1650	S
8	1681	1650	F
9	1300	1650	S
10	1489	1100	F

[a]S = success, F = failure.

Monte Carlo Simulation

❖ Monte Carlo Simulation

- Based on the observation that in five cases out of 10 the load is not supplied, the LOLP is estimated to be 0.5. If another sample of 10 simulations was taken or a larger sample was considered, the resulting probability would most likely be different.

- In fact, 100, 1000, 10,000, and up to one million simulations were performed, and the resulting loss of load probability is plotted in Fig. 13.3. The solid line interpolates the LOLP from the computed values.

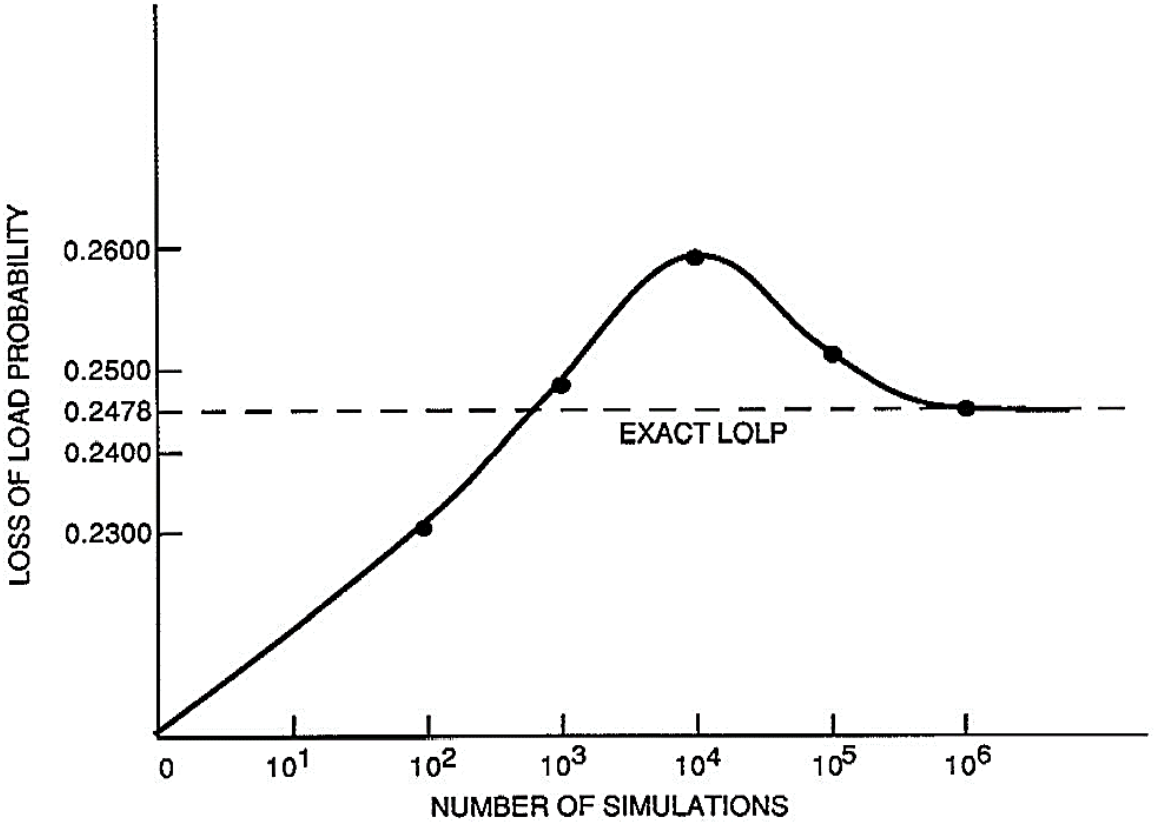


Figure 13.3 LOLP as a function of the number of simulations.

Monte Carlo Simulation

❖ Monte Carlo Simulation

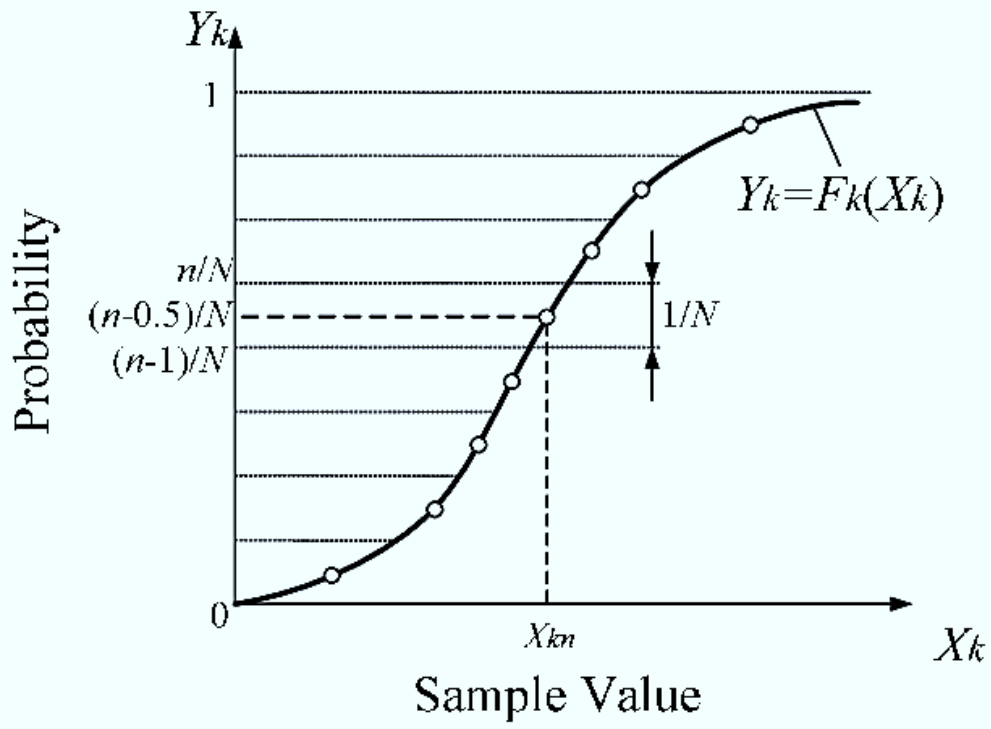
- It needs a large number of repetitions for the convergence.
- If the number of repetitions is large, it tends to the average amount of repetitior
- It is one of the time-consuming methods for generating different scenarios.
- It may not be considered a suitable solution for practical problems.



Latin Hyperbolic Sampling (LHS)

❖ Latin Hyperbolic Sampling (LHS)

- LHS is one of the proper methods for scenario generation in the stochastic problems.
- In this method for scenario number of Ns, the scale of cumulative probability (0 to 1.0) is divided to Ns non-overlapping intervals with the same lengths.
- A sample can be then selected randomly in the corresponding interval, or midpoint of the interval can also be assumed as the representative of the interval. Sampling selection is repeated to extract values of all intervals, thus the input probability distribution would be recreated using the extracted values of intervals.



- In this method, since the small number of scenarios affects the accuracy of the solution, the number of scenarios should be selected carefully.

Generation of Random Numbers

❖ Generation of Random Numbers

- Every Monte Carlo study requires generation of the appropriate values of the random variables in accordance with the respective prescribed probability distribution.

- This can be accomplished systematically for each variable by first generating a uniformly distributed random number between O and 1.o and then, through appropriate transformations, obtaining the corresponding random number with the specified probability distribution.

- For a given cumulative probability value u, we have sought the value of x such that Fx(x) = u. The value of variable X, therefore, is obtained by evaluating an inverse function:

$$x = F_x^{-1}(u)$$

Generation of Random Numbers

❖ Uniformly Distributed Random Numbers

Suppose now that u is a value of the *standard uniform variate*, U, with the CDF defined as (see Example 3.23)

$$F_U(u) = \begin{cases} 0 & \text{for } u < 0 \\ u & \text{for } 0 \le u < 1 \\ 1.0 & \text{for } 1 \le u \end{cases}$$

Therefore, if u is a value of U, the corresponding value of X obtained through Eq. 13.1 will have the cumulative probability

$$P(X \le x) = P[F_X^{-1}(U) \le x] = P[U \le F_X(x)] = F_U[F_X(x)] = F_X(x)$$

which means that if $(u_1, u_2, \ldots, u_n)$ is a set of values from U, the corresponding set of values obtained through Eq. 13.1, that is,

$$x_i = F_X^{-1}(u_i), \qquad i = 1, 2, \ldots, n \tag{13.2}$$

will have the desired CDF $F_X(x)$. The relationship between u and x may be seen graphically in Fig. 13.4.

Generation of Random Numbers

❖ Uniformly Distributed Random Numbers

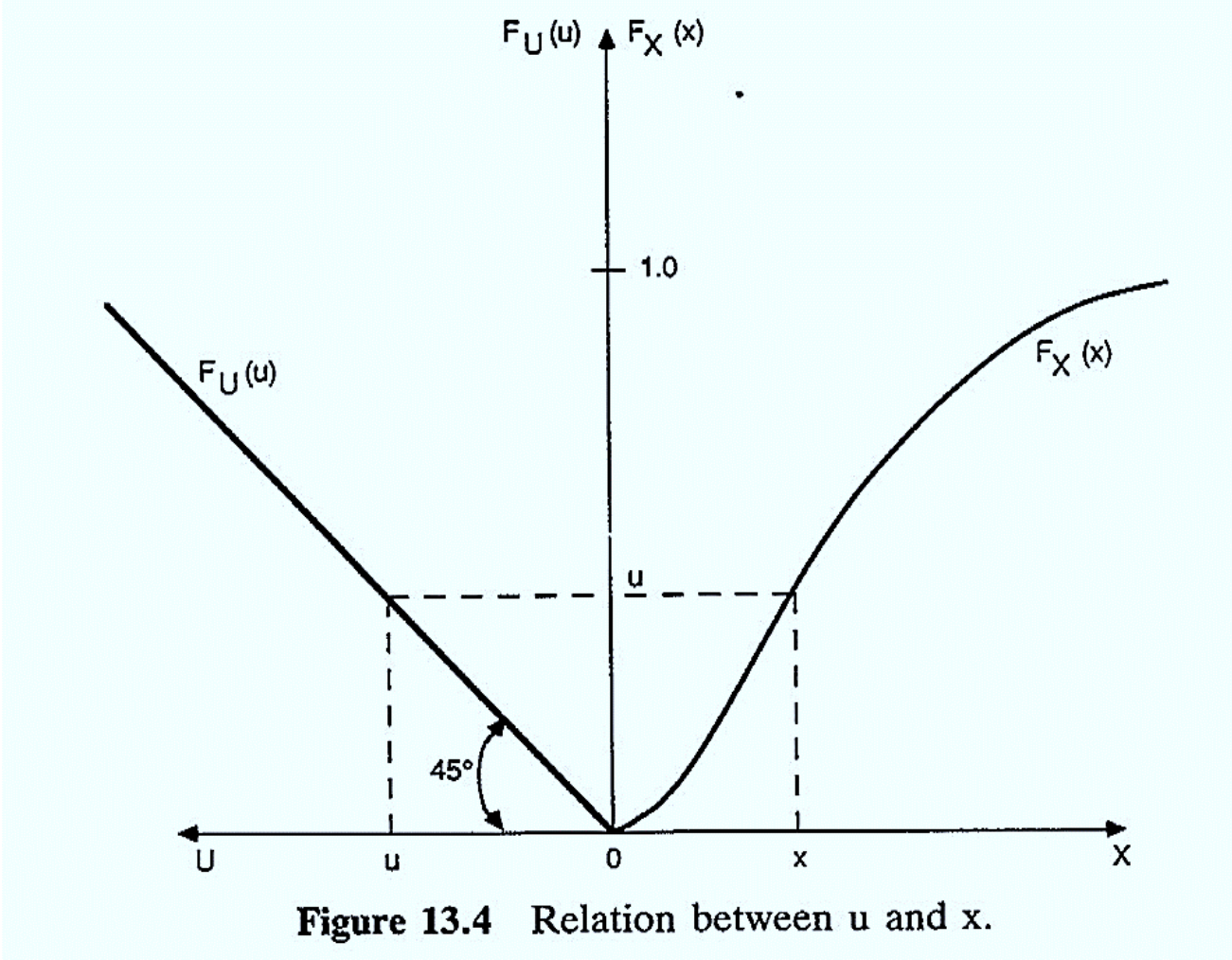


Figure 13.4 Relation between u and x.

Generation of Random Numbers

❖ Exponential Distribution

- The CDF of an exponential distribution function is given by:

$$F_X(x) = 1 - e^{-\lambda x}, \qquad x \geq 0$$

- Taking the inverse of this function, we can obtain the exponentially distributed random numbers from:

$$x_i = -\frac{1}{\lambda}\ln(1-u_i) = -\frac{1}{\lambda}\ln u_i$$

- We have utilized here the fact that if u is uniformly distributed, then so is 1-u.

Generation of Random Numbers

❖ Extreme Value Distribution

- The distribution function is given by:

$$F_X(x) = \exp[-e^{-\alpha(x-\beta)}]$$

- where beta is the most probable value of X and where a is the shape parameter.

- Taking the inverse of this function, we can obtain the random numbers with Type I extreme value distribution from:

$$x_i = \beta - \frac{1}{\alpha} \ln\left(\ln \frac{1}{u_i}\right)$$

Use Normal distribution to compute it again.

Generation of Random Numbers

❖ Formula for other distributions

TABLE 13.2 Random Number Generators for Some Common Distributions[a]

Distribution (parameters)	Formula	Comments
Normal (μ, σ)	$x_1 = \mu + \sigma\sqrt{-2 \ln u_1} \cos 2\pi u_2$ $x_2 = \mu + \sigma\sqrt{-2 \ln u_1} \sin 2\pi u_2$	x_1 and x_2 are a pair of random numbers from a normal distribution $N(\mu, \sigma)$.
Log-normal (λ, ξ)	$y = e^x$	Where x is the normal variate from the distribution $N(\lambda, \xi)$.
Beta (q, r)	$x = \dfrac{u_1^{1/q}}{u_1^{1/q} + u_2^{1/r}}$	u_1 and u_2 must satisfy the relation $u_1^{1/q} + u_1^{1/r} \leq 1$.
Gamma (λ, k)	$x = -\dfrac{1}{\lambda} \sum_{i=4}^{k'+3} \ln u_i - \dfrac{1}{\lambda} \ln u_3 \dfrac{u_1^{1/k}}{u_1^{1/k} + u_2^{1/(1-k)}}$	$k' \geq 1$ is an integer part of k. A total of $k' + 1$ random numbers must be generated to obtain one gamma distributed random number.

[a] u_i and u_2 are two random numbers from a standard uniform distribution.

Generation of Random Numbers

❖ Exercise:

- ❖ Obtain the formula for Rayleigh, Weibull, Normal distributions and verify it using Matlab

Generation of Random Numbers

❖ Exercise2:

- **Suppose that the PDF of X and Y is known, obtain the PDF of Z using MCS:**
- Z=2*X^2*Y*SIN(X*Y)+Y^0.5
- X ~ N(10,4)
- Y ~ U(3,9)

Generation of Random Numbers

❖ Discrete distribution

- In the case of a discrete random variable the inverse transform procedure has to be performed numerically.

- If u is a random number from a standard uniform distribution, the corresponding random number x, from among the possible values Xi, x2, ... of the random variable X is the one which satisfies the relationship:

$$F_X(x_{i-1}) < u \leq F_X(x_i)$$

- The statistical information available for power system components is most often represented in the form of a histogram. The random numbers corresponding to the underlying distribution function can be obtained in this case without actually developing the distribution function itself.

Generation of Random Numbers

❖ Discrete distribution

- Suppose that there are J intervals in the histogram with the ith interval having n, observations. Consider the ith interval of the histogram bounded by the values a,_1 and a, of the random variable X. Denoting by x, the representative value of X in this interval (e.g., one located in the middle of this interval; i.e., x, = (a,_1 + a,)/2), we can represent the distribution function of X by a stepwise cumulative frequency function:

$$F_X(x_i) = \begin{cases} 0, & x_i < a_0 \\ \frac{1}{N} \sum_{j=1}^{i} n_j, & a_0 \leq x_i < a_J \\ 1, & x_i \geq a_J \end{cases}$$

Generation of Random Numbers

❖ Discrete distribution

- As illustrated in Fig. 13.5, where N is the total number of observations. Substituting the above equation and multiplying all sides by N, we obtain:

$$0 < Nu - \sum_{j=1}^{i-1} n_j \leqq n_i$$

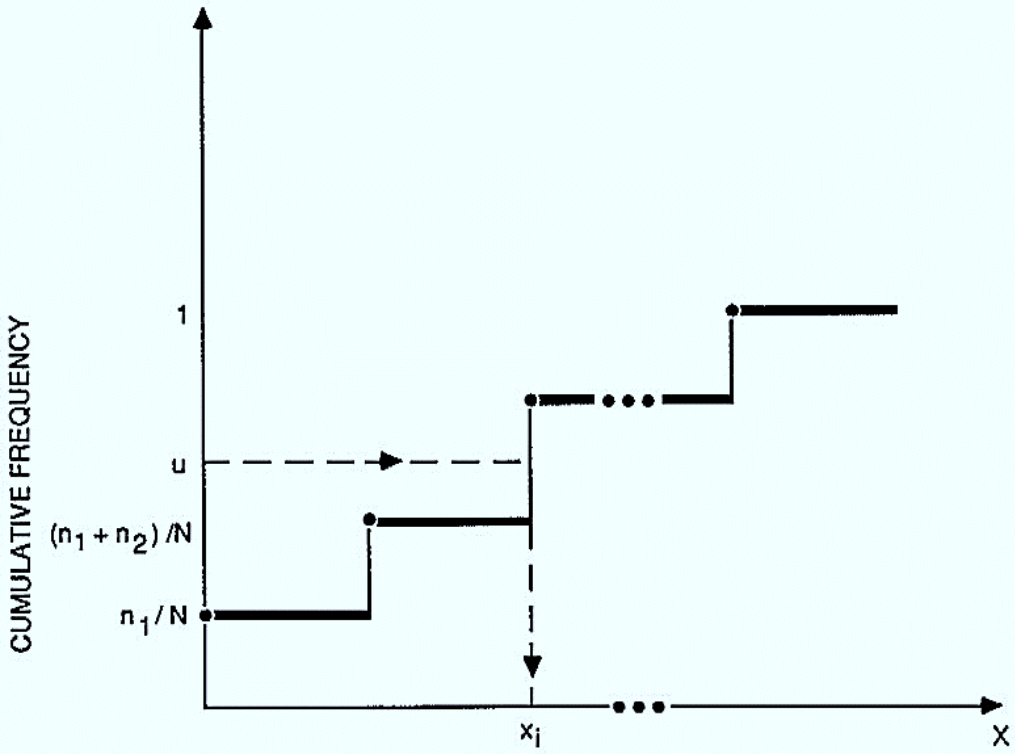


Figure 13.5 Random number from a cumulative frequency curve.

Generation of Random Numbers

❖ Discrete distribution

Example 13.3: An analysis of cable loadings in the month of August in downtown Toronto was performed. A histogram showing the number of occurrences of a particular MVA daily peak demand divided by the monthly peak was obtained as shown in Fig. 13.6.

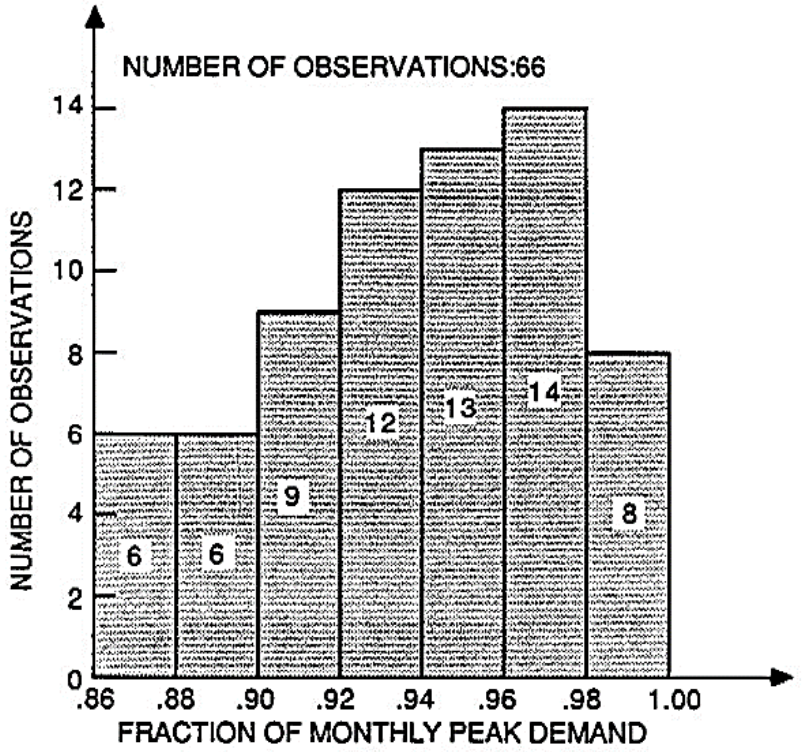


Figure 13.6 Histogram of the daily peak load for the month of August.

Generation of Random Numbers

❖ Discrete distribution

- Suppose that a value of u = 0.6000 has been generated. Determine the corresponding random value of the peak loading of the cables. We selected (quite arbitrarily) the upper bound of each interval as the representative value X;. The values of the middle side of the inequality 13.6 for increasing values of i are as follows:

- This means that the per-unit peak load value of 0.96 should be used with the selection of the random number u = 0.60.

i	x_i	$Nu - \sum_{j=1}^{i-1} n_j$	n_i
1	0.88	40.8	6
2	0.90	34.8	6
3	0.92	28.8	9
4	0.94	19.8	12
5	0.96	7.8	13

Generation of Random Numbers

❖ Approximation of binomial distribution

- As we indicated earlier, the inverse transform procedure may become time consuming when the number of intervals is large, or it may become time consuming in the case of the binomial distribution, when n is large.

Fortunately, the two most commonly used theoretical distributions in power system applications, the binomial and Poisson, can be approximated by normal distributions when certain conditions are met. For $np > 10$ and $p \geq 0.5$, or $n(1-p) > 10$ and $p < 0.5$, it can be shown that a binomial random variable X approaches a normal variate with mean $(np - 0.5)$ and standard deviation $np(1-p)$. Therefore, to generate a value x from a binomial distribution, a value y may first be generated from the normal distribution $N[np - 0.5, np(1-p)]$, and then we examine the following:

(i) If $y \leq 0$, set $x = 0$.
(ii) If $y \geq n$, set $x = n$.

Generation of Random Numbers

❖ Approximation of binomial distribution

(iii) If $0 < y < n$, round off y to the nearest integer and set x equal to that integer.

In the case of the Poisson distribution, if λ is large, say $\lambda > 10$, we can apply a normal distribution with mean $(\lambda - 0.5)$ and standard deviation $\sqrt{\lambda}$. As in the case of the binomial distribution, first we generate a normal variate y and then we examine the following:

(i) If $y \leq 0$, set $x = 0$.
(ii) If $y > 0$, round off y to the nearest integer and set x equal to that integer.